ESSENTIALS OF PLANE TRIGONOMETRY

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PREFACE

THE purpose of this book is to present in as simple a manner as possible, yet without lacking rigor, the material for a moderately short course in Plane Trigonometry. Since no previous knowledge of logarithms is assumed, the first chapter is devoted to the study of exponents and logarithms.

In the discussion of the trigonometric functions of an angle, those of an acute angle are defined first, and there are sufficient applications to familiarize the student with the subject before the functions of a general angle are introduced. The author has endeavored to show that the *process* used in defining the functions of a general angle is exactly the same as the one used for an acute angle; and that, moreover, when a general angle happens to be acute, the definitions are identical.

As pointed out in Chapter V, the author feels that a knowledge of the law of sines and the law of cosines and an intelligent application of the tables of squares and square roots are adequate tools for the solution of the oblique triangle. However, in deference to usage, the chapter "Supplementary Topics" contains the usual law of tangents and the *r* formulas, as well as a discussion of circular measure.

The fundamental formulas and identities are placed in a single chapter, and abundant problem material on solving identities is included in the text.

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Chapter I

LOGARITHMS

1. Exponents. Since a knowledge of the theory of exponents is essential for a clear understanding of logarithms, we shall review briefly that theory. By 5^3 we mean

$$5 \times 5 \times 5.$$

By a^3 we mean

$$a \times a \times a.$$

By a^m, provided m is a positive integer, we mean

$$a \times a \times a \ . \ . \ . \text{ to } m \text{ factors.}$$

We call a the *base* and m the *exponent.*

Consider the product of $5^3 \times 5^4$.

By this we mean

$$(5 \times 5 \times 5)(5 \times 5 \times 5 \times 5)$$

or

$$5 \times 5 \ . \ . \ . \text{ to seven factors}$$

or

$$5^7.$$

Similarly, $a^m \cdot a^n$ equals

$$(a \times a \ . \ . \ . \text{ to } m \text{ factors})(a \times a \ . \ . \ . \text{ to } n \text{ factors}).$$

Or:

$$a \times a \ . \ . \ . \text{ to } m + n \text{ factors} = a^{m+n}.$$

It is evident that this process gives the law:

The product of two or more quantities with a common base equals a quantity with the same base and an exponent equal to the sum of the exponents of the various quantities.

Now consider

$$\frac{5^5}{5^3}.$$

By this we mean

$$\frac{\cancel{5} \times \cancel{5} \times \cancel{5} \times 5 \times 5}{\cancel{5} \times \cancel{5} \times \cancel{5}} = 5 \times 5 = 5^2;$$

that is, the number 5 with an exponent equal to the difference of the exponent of the numerator and that of the denominator:

$$5^{5-3}.$$

Or, in general, if m is greater than n, then

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{\cancel{a} \times \cancel{a} \ldots \text{to } m \text{ factors}}{\cancel{a} \times \cancel{a} \ldots \text{to } n \text{ factors}} \\ &= a \times a \ldots \text{to } m - n \text{ factors} \\ &= a^{m-n}. \end{aligned}$$

Hence we have the law:

The quotient of two quantities with a common base equals a quantity with the same base and an exponent equal to the difference of the exponent of the numerator and that of the denominator.

Now consider

$$\frac{5^3}{5^5}.$$

$$\frac{\cancel{5} \times \cancel{5} \times \cancel{5}}{\cancel{5} \times \cancel{5} \times \cancel{5} \times 5 \times 5} = \frac{1}{5^2}.$$

However, if the above law is to hold,

$$\frac{1}{5^2} = 5^{3-5} = 5^{-2}.$$

It is quite apparent, therefore, that 5^{-2} does not imply

$$5 \times 5 \ldots \text{to } -2 \text{ factors}$$

(which is meaningless), but is a symbol for

$$\frac{1}{5^2}.$$

Similarly, if m is less than n, then

$$\frac{a^m}{a^n} = a^{m-n}$$

is a symbol for

$$\frac{1}{a^{n-m}}.$$

Hence we shall define a^{-m}, when m is greater than 0 (that is, m positive) to be equal to

$$\frac{1}{a^m}.$$

In particular, consider

$$\frac{a^m}{a^n},$$

where m equals n; that is,

$$\frac{a^m}{a^m}.$$

By the above law this equals

$$a^{m-m} = a^0.$$

But we know that

$$\frac{a^m}{a^m} = 1.$$

Therefore, for consistency, we define a^0 to be equal to 1.

Now let us see what we mean by a quantity with a *fractional* exponent. First, what do we mean by the symbol $\sqrt{3}$? We mean that quantity which multiplied by itself gives 3. Let $\sqrt{3} = 3^x$, and determine x. We have

$$3^x \cdot 3^x = 3,$$

or

$$3^{2x} = 3^1.$$

Then

$$2x = 1.$$

Therefore:

$$x = \tfrac{1}{2}.$$

Hence we define $3^{1/2}$ to equal

$$\sqrt{3}.$$

In general, we define $a^{m/n}$ as follows:

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

One fundamental law of exponents remains to be considered. Take $(5^3)^2$. By our first law,

$$(5^3)^2 = (5^3)(5^3) = 5^6.$$

Similarly,

$$\begin{aligned} (a^m)^n &= (a^m)(a^m) \ \ldots \text{ to } n \text{ factors} \\ &= a^{m+m+\ \ldots \text{ to } n \text{ terms}} \\ &= a^{nm} \\ &= a^{mn}. \end{aligned}$$

From these computations we have the law:

If a quantity with a given base and exponent is raised to a power, the result is a quantity with the same base and an exponent equal to the product of the two exponents.

The application of all three laws is allowable for fractional exponents, positive and negative, as well as for positive and negative integral exponents.

Problems

1. Find the value of: $(243)^{-3/5}$; $\sqrt[12]{8^4}$; $(-\frac{1}{125})^{2/3}$; $(742)^0$; $(x + 6y - 3)^0$; $1^{-7/9}$.

2. Express with positive exponents only:

$$\frac{5^{-2}x^{2/3}y^{-3/8}x^{-1/6}}{125^{-1/3}x^2y^{-2}}.$$

3. Express without any denominator:

$$\frac{2x^{-1/2}y^{2/3}x^{3/5}}{3^{-2}y^2x^{-2}}.$$

4. Solve: (*a*) $x^{2/3} = 4$; (*b*) $x^{4/5} = 1$.

5. Solve:

(*a*) $7^x = 49$.	(*c*) $27^x = 3$.	(*e*) $27^x = \frac{1}{9}$.
(*b*) $2^x = 8$.	(*d*) $27^x = 9$.	(*f*) $27^x = \frac{1}{3}$.

6. Solve:

(*a*) $10^x = 1000$.	(*d*) $10^x = 1$.	(*g*) $10^x = .001$.
(*b*) $10^x = 100$.	(*e*) $10^x = .1$	(*h*) $10^x = .0001$.
(*c*) $10^x = 10$.	(*f*) $10^x = .01$.	(*i*) $10^x = .00001$.

7. Solve: $2^{2x} - 6 \cdot 2^x + 8 = 0$.

8. Solve: $9 \cdot 3^{2x} - 244 \cdot 3^x + 27 = 0$. (Answer: $x = -2$ or 3.)

9. Simplify:

$$\left(\frac{2}{5}\right)^{-3} + 16^{-3/4} + \frac{5}{2^{-2}} + (-2)^{-2}.$$

10. Simplify: $(3^{n+2} + 3 \cdot 3^n) \div (9 \cdot 3^{n+2})$.

2. Definition of a logarithm. Consider: 7^2 equals 49. Observe that 2 is the exponent of the power to which 7 must be raised to give 49. We shall now, by means of a definition, write this expression in another form. We define the *logarithm* of 49 to the base 7 to be equal to 2. Or, expressed in symbols,

$$\log_7 49 = 2.$$

Similarly, since 8^2 equals 64, we have

$$\log_8 64 = 2;$$

and since 2^3 equals 8, we have

$$\log_2 8 = 3.$$

In general, if b^x equals N, we have

$$\log_b N = x;$$

and the general definition:

The *logarithm* of a number N to the base b is the exponent

of the power to which the base b must be raised to give the number N.

Example

Find $\log_2 64$.

Let $\log_2 64 = x.$

Then
$$2^x = 64,$$
$$2^x = 2^6,$$
$$x = 6.$$

Therefore: $\log_2 64 = 6.$

Problems

Find:

1. $\log_3 27$.	**5.** $\log_4 64$.	**9.** $\log_{10} 1000$.	**13.** $\log_{10} .1$.
2. $\log_{27} 3$.	**6.** $\log_9 27$.	**10.** $\log_{10} 100$.	**14.** $\log_{10} .01$.
3. $\log_3 \frac{1}{27}$.	**7.** $\log_{27} 9$.	**11.** $\log_{10} 10$.	**15.** $\log_{10} .001$.
4. $\log_8 16$.	**8.** $\log_{125} 5$.	**12.** $\log_{10} 1$.	**16.** $\log_{10} .0001$.

3. Laws of logarithms. There are three important laws of logarithms which are immediate consequences of the three laws of exponents and the definition of a logarithm.

The first law is, given the numbers M and N:

$$\log_b (MN) = \log_b M + \log_b N.$$

Proof

Let
$$\log_b M = x,$$
$$\log_b N = y.$$

Then
$$b^x = M,$$
$$b^y = N.$$

(Since we are interested in the logarithm of the product MN, we form that product.)

Hence: $MN = b^x \cdot b^y;$

or: $MN = b^{x+y}.$

In logarithmic form,

$$\log_b MN = x + y$$
$$= \log_b M + \log_b N.$$

The second law is:

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

Proof

As before, let $\log_b M = x,$

$\log_b N = y.$

Then $b^x = M,$

$b^y = N.$

Hence: $\dfrac{M}{N} = \dfrac{b^x}{b^y};$

or: $\dfrac{M}{N} = b^{x-y}.$

In logarithmic form,

$$\log_b \frac{M}{N} \quad x - y$$

$$= \log_b M - \log_b N.$$

The third law is:

$$\boxed{\log_b M^p = p \log_b M}$$

Proof

Let $\log_b M = x.$

Then $b^x = M.$

Hence: $M^p = b^{px}.$

In logarithmic form,

$$\begin{aligned} \log_b M^p &= px \\ &= p \log_b M. \end{aligned}$$

We state these laws as follows:

Law 1. *The logarithm of the product of two or more numbers to the same base is the sum of the logarithms of the respective numbers.*

Law 2. *The logarithm of the quotient of two numbers to the same base is the difference of the logarithm of the numerator and the logarithm of the denominator.*

Law 3. *The logarithm of a number raised to a power is the product of the exponent of the power and the logarithm of the number.*

It is quite apparent that law 3 handles the logarithm of the root of a number.

Problems

1. Write in expanded form:

$$(a)\ \log_b \frac{a^2bc}{2\sqrt{d}}; \qquad (b)\ \log_b \frac{3x^3y^{1/2}}{x^2}.$$

2. Write in contracted form: $2 \log_b x + \frac{1}{2} \log_b y - \log_b xy^2$.

3. Determine which of the following symbols equals $2 \log_b x$:

(a) $\log_b{}^2 x$. $\qquad$ (b) $\log_b x^2$. $\qquad$ (c) $(\log_b x)^2$

4. Common logarithms. For numerical computation it is desirable that a universal base be employed, and the most convenient base to use is the base 10. Logarithms with base 10 are called *common*, or *Briggs*, logarithms. (It may be stated, however, that in higher mathematics the base used is the irrational number $e = 2.71828 \ldots$.) It is understood that in this text the base is 10 unless otherwise stated.

Let us construct a miniature table of logarithms by considering various powers of 10.

$$\begin{aligned}
10^4 &= 10{,}000; &&\text{hence, } \log 10{,}000 = 4\\
10^3 &= 1000; &&\text{hence, } \log 1000 = 3\\
10^2 &= 100; &&\text{hence, } \log 100 = 2\\
10^1 &= 10; &&\text{hence, } \log 10 = 1\\
10^0 &= 1; &&\text{hence, } \log 1 = 0\\
10^{-1} &= .1; &&\text{hence, } \log .1 = -1
\end{aligned}$$

$$10^{-2} = .01; \quad \text{hence, } \log .01 = -2$$
$$10^{-3} = .001; \quad \text{hence, } \log .001 = -3$$
$$10^{-4} = .0001; \quad \text{hence, } \log .0001 = -4$$

Now suppose we wish the logarithm of a number not given here, say 386. Obviously since 386 is between 100 and 1000, the log of 386 is between 2 and 3; that is, 2 plus a fraction less than 1. This fraction, in decimal form, is given in the tables; it is found to be .5866. Hence:

$$\log 386 = 2.5866.$$

Or:

$$10^{2.5866} = 386.$$

Now suppose we wish the log of 38.6. Since this number is between 10 and 100, its log is 1 plus a fraction; from the tables we find the fraction to be the same as that noted above; that is, .5866. Hence:

$$\log 38.6 = 1.5866.$$

From these computations we draw the following conclusions: The logarithm of a number is composed of two parts, an integral part and a fractional or decimal part. We call the integral part, the *characteristic*, and the fractional part, the *mantissa*. For a given number, a shift in the position of the decimal point changes the characteristic but does not change the mantissa.

The characteristic of the log of 386 was found to be 2. It is quite apparent that the characteristic of the log of *every* number between 100 and 1000 is 2; that is, the characteristic of the log of every number with three digits to the left of the decimal point is 2. And it is not difficult to see that the characteristic of the log of a number greater than 1 is always numerically *one less* than the number of digits to the left of the decimal point.

Now let us consider the logs of positive numbers less than 1. Consider log .00386. From our miniature table, since .00386 is between .001 and .01, log .00386 is between -3

and -2; that is, -3 plus a positive fraction less than 1, or, -3 plus .5866—which we write as $\bar{3}.5866$. Hence:

$$\log .00386 = \bar{3}.5866.$$

The characteristic is -3, a negative number and numerically *one more* than the number of zeros immediately to the right of the decimal point. In general the characteristic of the log of a positive number less than 1 is negative and numerically one more than the number of zeros immediately to the right of the decimal point. In the latter instance we may always, if we prefer, assume the decimal point to be in a convenient position, compute the characteristic of the log of that number, and then shift the decimal point to its proper position and revise the characteristic by counting.

5. Use of the logarithmic tables. Suppose we wish log 726. The line from our tables is

N	0	1	2	3	4	5	6	7	8	9
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627

The characteristic is 2. The mantissa is found as follows: Look under N for the first two digits, 72. Then, since our third digit is 6, our mantissa is on the line of 72 and directly under 6. It is .8609. Hence:

$$\log 726 = 2.8609.$$

Or, suppose we know that the log of a number N is 1.8591 and we wish to find N. We look for the number corresponding to the mantissa 8591 and observe it is 723. Then, since the characteristic is 1, the number N must have two digits to the left of the decimal point. Therefore:

$$N = 72.3.$$

The process of finding a number N when log N is given is obviously the inverse process of finding a log. N we call the *antilogarithm*. In the process of finding an antilogarithm, the characteristic simply indicates the position

of the decimal point. And it is apparent that, for a positive or zero characteristic, the resulting antilogarithm has *one more* digit to the left of the decimal point than the numerical value of the characteristic; and that, for a negative characteristic, the antilogarithm has *one less* zero immediately to the right of the decimal point than the numerical value of the characteristic. Thus, if log N equals $\bar{3}.8591$, then

$$N = .00723.$$

Problems

1. Find the logarithms of the following:

(*a*) 382.	(*e*) 382×10^{-6}.	(*i*) 100.
(*b*) 3.82.	(*f*) 4.61.	(*j*) 10^8.
(*c*) 38,200.	(*g*) .000279.	(*k*) .0243.
(*d*) 382×10^6.	(*h*) .00963.	(*l*) 10^{-2}.

2. Find the antilogarithms of the following:

(*a*) 2.4829.	(*d*) 1.7050.	(*g*) $\bar{3}.9624$.
(*b*) $\bar{1}.4829$.	(*e*) 2.8808.	(*h*) .8306.
(*c*) 0.4829.	(*f*) 1.9763.	(*i*) $\bar{2}.0000$.

6. Interpolation. The logarithms of all numbers of three digits, preceded or followed by any number of zeros, can be found in our tables by the process described in Section 5. We shall now show how, by *interpolation*, we can find logarithms of numbers of four digits.

Consider log 72.63. The tables give us log 72.60 and log 72.70. Hence we have:

$$\log 72.60 = 1.8609$$
$$\log 72.63 = (?)$$
$$\log 72.70 = 1.8615$$

We reason as follows: The number 72.63 is three-tenths of the way from 72.60 to 72.70. Hence, log 72.63 is three-tenths of the way from log 72.60 to log 72.70; that is, three-tenths of the way from 1.8609 to 1.8615. Subtracting 1.8609 from 1.8615, we find that the *distance*

between the two is 6 units. The desired logarithm is therefore:

$$\begin{aligned} 1.8609 + .3 \times 6 \text{ units} &= 1.8609 + 1.8 \text{ units} \\ &= 1.8609 + 2 \text{ (approx.) units} \\ &= 1.8611. \end{aligned}$$

Hence:

$$\log 72.63 = 1.8611.$$

Similarly,

$$\log 726.3 = 2.8611,$$

and so on.

Find log 384.2.

$$\begin{aligned} \log 384.0 &= 2.5843 \\ \log 384.2 &= (?) \\ \log 385.0 &= 2.5855 \end{aligned}$$

Log 384.2 is two-tenths of the way from 2.5843 to 2.5855. The distance between them is 12 units.

$$\begin{aligned} .2 \times 12 &= 2.4 \\ &= 2 \text{ (approx.).} \end{aligned}$$

Hence:

$$\log 384.2 = 2.5845.$$

In like manner, interpolation is used in finding antilogarithms of numbers not given in the tables. Consider antilog 2.7463. We look in the tables for the two mantissas nearest to 7463, and find 7459 and 7466. Hence, disregarding temporarily the correct position of the decimal point, we have:

$$\begin{aligned} \text{antilog } 7459 &= 5570 \\ \text{antilog } 7463 &= (?) \\ \text{antilog } 7466 &= 5580 \end{aligned}$$

As before, antilog 7463 must be the same proportion of the way between 5570 and 5580 as 7463 is between 7459

and 7466. We first find what proportion this is. The distance from 7459 to 7466 is 7 units. The distance from 7459 to 7463 is 4 units. Hence, 7463 is four-sevenths of the way from 7459 to 7466. Hence, antilog 7463 is four-sevenths of the way from 5570 to 5580. Therefore:

$$\begin{aligned}\text{antilog } 7463 &= 5570 + \tfrac{4}{7} \times 10 \text{ units}\\ &= 5570 + \tfrac{40}{7} \text{ units}\\ &= 5570 + 6 \text{ (approx.) units}\\ &= 5576.\end{aligned}$$

Hence:

$$\text{antilog } 2.7463 = 557.6.$$

Problems

1. Find the logarithms of:

(*a*) 3.286.	(*d*) .0003428.	(*g*) .01111.
(*b*) 729.4.	(*e*) 82.37.	(*h*) .3263.
(*c*) 68.43.	(*f*) 42.94.	(*i*) .02438.

2. Find the antilogarithms of:

(*a*) 2.8531.	(*d*) $\bar{2}.8906$.	(*g*) $\bar{2}.2375$.
(*b*) 1.9276.	(*e*) $\bar{1}.8660$.	(*h*) .3770.
(*c*) 4.6081.	(*f*) 1.0200.	(*i*) .0964.

7. Applications of the laws of logarithms, and a few tricks.

Example 1

By logarithms, find:

$$\frac{(24.32)(6.431)}{76.47}.$$

By our first two laws of logarithms, the log of the required number N may be expressed as follows:

$$\log N = \log 24.32 + \log 6.431 - \log 76.47.$$

$$\log 24.32 = 1.3860$$

$$\log 6.431 = 0.8083$$

$$\begin{array}{lrl} \text{Therefore:} & \log \text{numerator} = & 2.1943 \\ & \log 76.47 = & \underline{1.8835} \\ \text{Subtracting,} & \log N = & .3108 \\ \text{Hence:} & N = & 2.045. \end{array}$$

No tricks are necessary in solving the above problem, but suppose the problem reads:

Find:

$$\frac{(24.32)(6.431)}{764.7}.$$

$$\begin{array}{lrl} \text{Then} & \log \text{numerator} = & 2.1943 \\ & \log 764.7 = & \underline{2.8835} \\ & \log N = & (?) \end{array}$$

The solution will involve a negative number. Subtracting correctly, we have:

$$\log N = \bar{1}.3108.$$

However, there is an easier process for solving this problem.

$$\begin{array}{lrl} \text{Write } 2.1943 \text{ as:} & & 12.1943 - 10. \\ \text{Then} & \log \text{numerator} = & 12.1943 - 10 \\ & \log 764.7 = & \underline{2.8835} \\ \text{Subtracting,} & \log N = & 9.3108 - 10 \\ \text{Or, as above,} & \log N = & \bar{1}.3108. \end{array}$$

Anticipating negative characteristics and writing them in this manner will be found a very useful practice. From this point in the text we shall write negative characteristics thus.

Suppose we wish

$$\log \frac{N_1}{N_2}$$

when $\log N_1$ equals $9.3241 - 10$, and $\log N_2$ equals $9.4762 - 10$. In this case, the following is the solution:

Write $\log N_1$ as: $\quad 19.3241 - 20.$

Then

$$\log N_1 = 19.3241 - 20$$
$$\log N_2 = 9.4762 - 10$$

Subtracting,

$$\log \frac{N_1}{N_2} = 9.8479 - 10.$$

Example 2

Find $\sqrt[3]{28.64}$ by logarithms.

By our third law of logarithms,

$$\begin{aligned}\log \sqrt[3]{28.64} &= \log (28.64)^{1/3}\\ &= \tfrac{1}{3} \log 28.64\\ &= \frac{\log 28.64}{3}\end{aligned}$$

Then

$$\begin{aligned}\frac{\log 28.64}{3} &= \frac{1.4570}{3}\\ &= .4856\tfrac{2}{3}\\ &= .4857.\end{aligned}$$

Therefore: $\text{antilog } .4857 = 3.060.$

Hence: $\sqrt[3]{28.64} = 3.060.$

Again no tricks are necessary. But consider the following:

Find $\sqrt[3]{.0002864}$.

Then

$$\begin{aligned}\frac{\log .0002864}{3} &= \frac{6.4570 - 10}{3}\\ &= 2.1523 - 3\tfrac{1}{3}.\end{aligned}$$

This answer is a bit confusing, however, since we are left with a fractional characteristic. The trouble lies in the fact that 10 is not exactly divisible by 3. Hence, the following is a better solution, since the answer can be handled more readily.*

* These two results may be shown to be the same, as follows:

$$\begin{aligned}2.1523 - 3\tfrac{1}{3} &= 2.1523 - 3.3333\\ &= 12.1523 - 10 - 3.3333\\ &= 8.8190 - 10.\end{aligned}$$

Write $\dfrac{\log .0002864}{3}$ as: $\dfrac{26.4570 - 30}{3}$.

Then $\dfrac{\log .0002864}{3} = 8.8190 - 10.$

Similarly, consider the following:

Solve:

$$\frac{\log N}{4} = \frac{8.2869 - 10}{4}.$$

Write $8.2869 - 10$ as: $38.2869 - 40$,
and so on.

Example 3

Find the amount to which \$100 will grow in 10 years if the interest is compounded semi-annually at 6 per cent.

If the interest were compounded annually, we should have at the end of one year:

$$\$100(1.06);$$

and at the end of 10 years:

$$\$100(1.06)^{10}.$$

If the interest is compounded semi-annually, we shall have at the end of six months:

$$\$100(1.03);$$

and at the end of one year:

$$\$100(1.03)^2;$$

that is,

$$\$100\left(1 + \frac{.06}{2}\right)^2.$$

Hence, at the end of 10 years we shall have:

$$\$100\left[\left(1 + \frac{.06}{2}\right)^2\right]^{10};$$

or:

$$\$100(1.03)^{20}.$$

Now, letting x equal the amount, we have the following:

$$\begin{aligned} \log x &= \log 100 + 20 \log 1.03 \\ &= 2.0000 + 20(.0128) \\ &= 2.2560. \end{aligned}$$

Therefore:

$$x = 180.3.$$

Our answer is:

$$\$180.30.$$

Example 4

Solve: $28.62^x = 684.9$.

Taking logs of each side, we have the following:

$$\begin{aligned} \log 28.62^x &= \log 684.9 \\ x \log 28.62 &= \log 684.9. \end{aligned}$$

Therefore:

$$\begin{aligned} x &= \frac{\log 684.9}{\log 28.62} \\ &= \frac{2.8356}{1.4567} \\ &= 1.947. \end{aligned}$$

Problems

1. Find:

(a) $\dfrac{(2.382)(69.84)}{(4236)(.02438)}$.

(b) $\sqrt[3]{41.72}$.

(c) $(3.461)^5$.

(d) $\sqrt[3]{\dfrac{(7.241)(62.86)}{296.3}}$.

(e) $\dfrac{(1.246)^2}{98.77}$.

(f) $(328.2)\sqrt[3]{.004691}$.

(g) $\dfrac{(1.286)^2(91.34)^{3/2}}{4.277}$.

(h) $\left(\dfrac{637.2}{9885}\right)^5$.

(i) $\dfrac{(72.39)(1.006)^2}{\sqrt[4]{879.3}}$.

(j) $\sqrt[7]{679.6}$.

2. Find the amount to which \$6000 will grow in 5 years if the interest is compounded quarterly at 4 per cent.

3. Solve: $4.287^x = 52.39$.

4. Find: $(12.43)^{\sqrt{3}}$.

5. Prove: $\log_a N = \log_b N \cdot \log_a b$.

6. Prove: $\log_a b \cdot \log_b a = 1$.

Chapter II

THE TRIGONOMETRIC FUNCTIONS

8. Angles. Suppose a line OX is revolved about the point O until it takes the position OP (Figure 1). An angle XOP is then generated. We call OX the *initial* side of the angle, and OP the *terminal* side. The angle may be denoted by the single letter θ. For the moment we shall consider θ as acute.

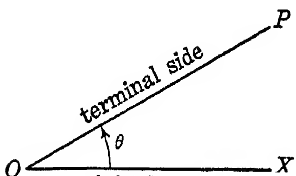

Figure 1.

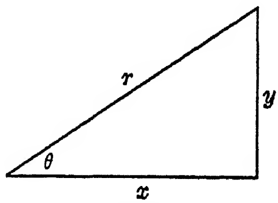

Figure 2

9. Trigonometric functions of an angle. Let us take any point on the terminal side and drop a perpendicular to the initial side (extended if necessary). A right triangle is then formed containing θ (Figure 2), with legs x and y, and hypotenuse r. Obviously the lengths x, y, and r are determined by the position of the point chosen, but the *ratios* of any two of the quantities x, y, r are unique for a given θ. There are six such ratios, called the six *trigonometric functions* of an angle θ; and they are defined as follows:

$$\text{sine } \theta = \sin\theta = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{cosine } \theta = \cos\theta = \frac{x}{r} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\text{tangent } \theta = \tan\theta = \frac{y}{x} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\text{cosecant } \theta = \csc \theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\text{secant } \theta = \sec \theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{cotangent } \theta = \cot \theta = \frac{x}{y} = \frac{\text{adjacent side}}{\text{opposite side}}$$

From these definitions and the law of Pythagoras, all six functions of an acute angle θ can be found if any one function is known. For example, suppose we are given

$$\sin \theta = \frac{}{5}.$$

Since

$$\frac{y}{r} = \frac{3}{5},$$

construct a right triangle with $y = 3$ and $r = 5$, as in Figure 3. Then

$$\begin{aligned} x^2 &= r^2 - y^2 \\ &= 25 - 9 \\ &= 16. \end{aligned}$$

Therefore:

$$x = 4.$$

Then we have:

$$\sin \theta = \frac{3}{5}.$$

$$\cos \theta = \frac{4}{5}.$$

$$\tan \theta = \frac{3}{4}.$$

$$\csc \theta = \frac{5}{3}.$$

$$\sec \theta = \frac{5}{4}.$$

$$\cot \theta = \frac{4}{3}.$$

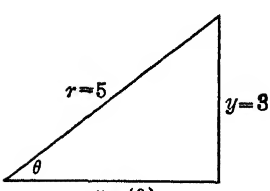

Figure 3.

Problems

1. Given Figure 2 (see page 21):

(*a*) Express x in terms of $\cos\theta$ and y or r.
(*b*) Express y in terms of $\cot\theta$ and x or r.
(*c*) Express r in terms of $\sin\theta$ and x or y.
(*d*) Express r in terms of $\csc\theta$ and x or y.

2. Given a right triangle with hypotenuse c, opposite side b, adjacent side a; find all functions of θ for the following:

(*a*) $a = 3, b = 4, \ c = 5.$
(*b*) $a = 5, b = 12, c = 13.$
(*c*) $a = 1, b = 2, \ c = \sqrt{5}.$
(*d*) $a = 2, b = 5, \ c = \sqrt{29}.$

3. Given $\cos\theta = \frac{12}{13}$; find $\sin\theta$.
4. Given $\csc\theta = \frac{5}{3}$; find $\tan\theta$.
5. Given $\tan\theta = 1$; find $\cot\theta$.
6. Given $\cot\theta = \sqrt{3}$; find $\sin\theta$.
7. Given $\sin\theta = \dfrac{1}{\sqrt{2}}$; find $\sec\theta$.

10. Functions of 30°, 45°, 60°. The functions of 30°, 45°, and 60° can be found from geometric considerations.

We shall find first the functions of 45°. If $\theta = 45°$, then, from plane geometry, the right triangle in Figure 2 is isosceles, and $x = y$. Hence we have immediately:

$$\tan 45° = 1.$$

In the right triangle in Figure 4, $x = 1$ and $y = 1$; and we have $r = \sqrt{2}$. Consequently we have:

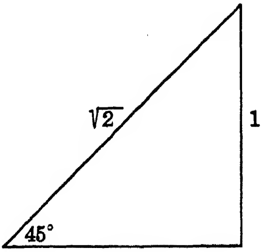

Figure 4.

$$\sin 45° = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{1}{\sqrt{2}}$$

tan 45° 1

$$\csc 45° = \sqrt{2}$$
$$\sec 45° = \sqrt{2}$$
$$\cot 45° = 1$$

We shall next find the functions of 30°. If $\theta = 30°$, as in Figure 5, then, from plane geometry (Figure 2), $r = 2y$. Hence:

$$\frac{y}{r} = \frac{1}{2}$$

and

$$\boxed{\sin 30° = \frac{1}{2}}$$

In the right triangle in Figure 5, $y = 1$ and $r = 2$; and we have $x = \sqrt{3}$. Consequently we have:

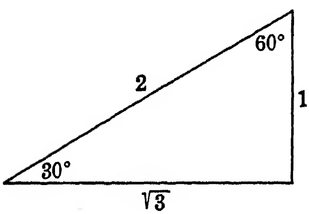

Figure 5.

$$\sin 30° = \frac{1}{2}$$
$$\cos 30° = \frac{\sqrt{3}}{2}$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$\csc 30° = 2$$
$$\sec 30° = \frac{2}{\sqrt{3}}$$
$$\cot 30° = \sqrt{3}$$

It is quite evident from the above explanations that the functions of 60° can be computed from Figure 5; but since the 60° angle may come at the base, we have added Figure 6. Consequently we have:

$$\sin 60° = \frac{\sqrt{3}}{2}$$
$$\cos 60° = \frac{1}{2}$$

$$\tan 60^\circ = \sqrt{3}$$

$$\csc 60^\circ = \frac{2}{\sqrt{3}}$$

$$\sec 60^\circ = 2$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}$$

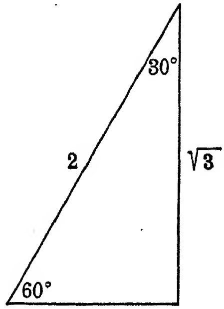

Figure 6.

The above eighteen functions are important because, since the functions are computed geometrically, no tables are necessary for computations with them and, hence, the functions furnish material for a host of problems in many branches of science. We recommend that the student memorize the two basic, simple relations that follow:

$$(1)\ \sin 30^\circ = \frac{1}{2}$$

$$(2)\ \tan 45^\circ = 1$$

since from these two relations the remaining sixteen relations can be computed readily.

11. Functions of $(90^\circ - \theta)$. In a comparison of the functions of 30° and 60°, it is observed that

$$\sin 30^\circ = \cos 60^\circ$$
$$\cos 30^\circ = \sin 60^\circ$$
$$\csc 30^\circ = \sec 60^\circ$$

and so on; that is, the functions of 30° are the corresponding co-functions of 60°. This conclusion suggests a relation between the functions of any acute angle θ and the corresponding co-functions of its complement.

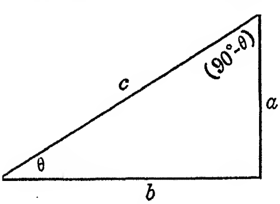

Figure 7.

We have, in Figure 7,

$$\sin \theta = \frac{a}{c} = \cos (90^\circ - \theta)$$

$$\cos \theta = \frac{b}{c} = \sin (90^\circ - \theta)$$

and so on.

Hence we have:

Theorem. *Any function of an acute angle θ equals the corresponding co-function of* $(90° - \theta)$, *and any function of* $(90° - \theta)$ *equals the corresponding co-function of* θ.

12. Tables of trigonometric functions. As illustrated in Section 10, the trigonometric functions of 30°, 45°, and 60° can be computed geometrically. By more advanced methods the trigonometric functions of other acute angles have been computed and tables have been made, the use of which is similar to that of logarithmic tables. Let us take two typical contiguous lines from our tables.

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
38° 00′	.6157	9.7893	.7813	9.8928	1.2799	.1072	.7880	9.8965	**52° 00′**
10′	.6180	9.7910	.7860	9.8954	1.2723	.1046	.7862	9.8955	50′
39° 00′									**51° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Degrees
	Cosine		Cotangent		Tangent		Sine		

These two lines give us the sine, cosine, tangent, and cotangent of 38°, 38° 10′, 52°, and 51° 50′, as well as the logs of these functions, a characteristic 9 standing for 9 − 10, and so on. The tables are so arranged that, for angles from 0° to 45°, we read from the left and from the top, working down. For angles from 45° to 90°, we read from the right and from the bottom, working up. For example, the sine of 38° is .6157, and the sine of 52° is .7880. Observe that the cosine of 52° is .6157, and the cosine of 38° is .7880. Sin 38° = cos 52° according to Section 11, since 38° and 52° are complementary angles. Similar results obtain for the tangent and the cotangent.

There are tables of secants and cosecants, but we shall not need them in our work.

The function tables in this book are so arranged, with intervals of 10 minutes, that interpolation is practically like interpolation in logarithms. Suppose, for example, we wish sin 38° 4′.

$$\begin{aligned} \sin 38^\circ \quad\ &= .6157 \\ \sin 38^\circ\ \ 4' &= (?) \\ \sin 38^\circ\ 10' &= .6180 \end{aligned}$$

Sin 38° 4′ is four-tenths of the way from .6157 to .6180. Therefore:

$$\begin{aligned} \sin 38^\circ\ 4' &= .6157 + .4(23) \text{ units} \\ &= .6157 + 9 \text{ units} \\ &= .6166. \end{aligned}$$

The tangent is similarly handled.

The cosine of an angle, however, *decreases* as the angle increases, and so is handled a bit differently. Consider cos 38° 4′.

$$\begin{aligned} \cos 38^\circ \quad\ &= .7880 \\ \cos 38^\circ\ \ 4' &= (?) \\ \cos 38^\circ\ 10' &= .7862 \end{aligned}$$

Cos 38° 4′ is four-tenths of the way from .7880 to .7862. Therefore:

$$\begin{aligned} \cos 38^\circ\ 4' &= .7880 - .4(18) \text{ units} \\ &= .7880 - 7 \text{ units} \\ &= .7873. \end{aligned}$$

Observe that we have *subtracted* instead of *added.*

The cotangent is handled similarly to the cosine.

Problems

1. Prove:

(*a*) $\sin 60^\circ = 2 \sin 30^\circ \cos 30^\circ.$
(*b*) $\sin 30^\circ = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

(c) $\tan 60^\circ = \dfrac{2 \tan 30^\circ}{1 - \tan^2 30^\circ}$.

(d) $\cos 30^\circ = \sqrt{1 + \cos 60^\circ}$

(e) $\tan 30^\circ = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

2. Solve for θ in the following equations:

(a) $\sin \theta = \cos \theta$.
(b) $\cos \theta = \sqrt{3} \sin \theta$.
(c) $\cos (90^\circ - \theta) = \frac{1}{2}$.

3. Find:

(a) $\sin 29^\circ\, 36'$.
(b) $\tan 53^\circ\, 27'$.
(c) $\sin 62^\circ\, 13'$.
(d) $\cos 40^\circ\, 42'$.
(e) $\cos 71^\circ\, 33'$.
(f) $\cot 42^\circ\, 26'$.

4. Find:

(a) $\log \sin 19^\circ\, 4'$.
(b) $\log \cos 40^\circ\, 17'$.
(c) $\log \tan 9^\circ\, 46'$.
(d) $\log \cot 48^\circ\, 4'$.
(e) $\log \sin 62^\circ\, 17'$.
(f) $\log \cos 72^\circ\, 8'$.

5. Find the angle (a) whose sine is .5883; and (b) whose cosine is .4072.

6. Find the angle (a) whose log tan is $9.7726 - 10$; and (b) whose log cot is 1.6000.

CHAPTER III

SOLUTION OF THE RIGHT TRIANGLE

13. Right triangle. If we are given a right triangle—and therefore know that one angle is 90°—and we are given, in addition, either of the other angles and any side, or any two sides, the triangle is determined uniquely. It is the purpose of this chapter to show how the trigonometric functions enable one to find the missing parts of a right triangle, when the above information is given. We call this process *solving the right triangle.*

Example 1

Given the right triangle ABC (Figure 8), with $c = 100$ and $A = 26° 14'$; solve the triangle.

$$B = 90° - 26° 14'$$
$$= 63° 46'$$

$$\frac{a}{c} = \sin A$$

$$\therefore a = c \sin A$$
$$= 100 \sin 26° 14'$$
$$= 100 \times .4420$$
$$= 44.20$$

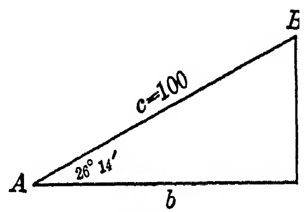

Figure 8.

$$\frac{b}{c} = \sin B$$

$$\therefore b = c \sin B$$
$$= 100 \sin 63° 46'$$
$$= 100 \times .8970$$
$$= 89.70$$

Or, we might have used the following solution:

$$\frac{b}{c} = \cos A$$

$$\begin{aligned}\therefore b &= c \cos A \\ &= 100 \cos 26^\circ\ 14' \\ &= 100 \times .8970 \\ &= 89.70\end{aligned}$$

Logarithms might have been used in this problem; however, having $c = 100$, as the multiplier, simplified the problem.

Example 2

Given the right triangle ABC; find a when $b = 382.6$ and $B = 70^\circ$.

$$\begin{aligned}A &= (90^\circ - B) \\ &= 20^\circ \\ \frac{a}{b} &= \tan A \\ \therefore a &= b \tan A\end{aligned}$$

With logarithms, $\log a = \log b + \log \tan A$

$$\begin{aligned}&= \log 382.6 + \log \tan 20^\circ \\ \log 382.6 &= 2.5828 \\ \log \tan 20^\circ &= \underline{9.5611 - 10}\end{aligned}$$

Adding, $\log a = 12.1439 - 10$

$$\begin{aligned}&= 2.1439 \\ \therefore a &= 139.3\end{aligned}$$

Example 3

Given the right triangle ABC; find A when $c = 389$ and $a = 202$.

$$\begin{aligned}\sin A &= \frac{a}{c} \\ \log \sin A &= \log a - \log c \\ &= \log 202 - \log 389 \\ \log 202 &= 12.3054 - 10 \\ \log 389 &= \underline{2.5899}\end{aligned}$$

Subtracting, $\log \sin A = 9.7155 - 10$

$$\therefore A = 31^\circ\ 18'$$

14. Angles of elevation and depression. The *angle of elevation* of an object above the eye of an observer is defined

as that angle which a line from the observer's eye to the object makes with a horizontal line. Roughly it is the angle through which the observer must *elevate* his eyes to see the object. Thus, in Figure 9, if the observer's eye is at E and the object is at O, the angle of elevation is θ.

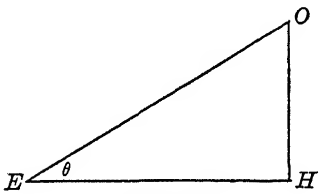

Figure 9.

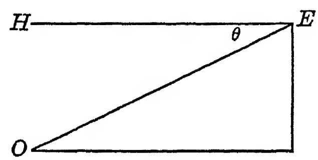

Figure 10.

If, on the other hand, the observer is above the object, the corresponding angle—that is, the angle through which the observer must *depress* his eyes—is called the *angle of depression* of the object. Thus, in Figure 10, if the observer's eye is at E and the object is at O, the angle of depression is θ.

Example

A tower stands on the shore of a river 207.2 feet wide (Figure 11). The angle of elevation of the top of the tower from the point on the other shore exactly opposite the tower is 44° 24′. Find the height of the tower.

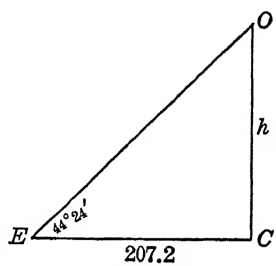

Figure 11.

We are given (assuming the observer's eye on the ground) $E = 44° 24'$, and $EC = 207.2$. We require h.

$$\frac{h}{207.2} = \tan 44° 24'$$

Hence $$\log h = \log 207.2 + \log \tan 44° 24'$$

$$\log 207.2 = 2.3164$$

$$\log \tan 44° 24' = 9.9909 - 10$$

Adding, $$\log h = 12.3073 - 10$$

$$= 2.3073$$

$$\therefore h = 202.9$$

The height of the tower, therefore, is 202.9 feet.

Problems

1. Solve the following right triangles, where $C = 90°$.

(*a*) $a = 2234$, $A = 36° 19'$.	(*f*) $c = 149.3$, $a = 26.24$.
(*b*) $b = 126.3$, $A = 58° 44'$.	(*g*) $c = 60.23$, $B = 68° 43'$.
(*c*) $a = 1406$, $b = 2173$.	(*h*) $c = 3204$, $b = 2062$.
(*d*) $a = 72.09$, $B = 24° 33'$.	(*i*) $a = 1.263$, $A = 80° 14'$.
(*e*) $c = 2434$, $A = 42° 26'$.	(*j*) $b = 2.304$, $A = 22° 46'$.

2. A ladder 41.24 feet long is so placed that it will reach a window 34.62 feet high on one side of a street. If it is turned over without moving its foot, the ladder will reach a window 20.28 feet high on the other side of the street. Find the width of the street.

3. A tower stands on the shore of a river 210.6 feet wide. The angle of elevation of the top of the tower from a point on the other shore directly opposite to the tower is $40° 52'$. Find the height of the tower.

4. A rope is stretched from the top of a building to the ground. The rope makes an angle of $52° 36'$ with the horizontal, and the building is 70 feet high. Find the length of the rope.

5. The top of a ladder 60.34 feet long rests against a wall at a point 46.23 feet from the ground. Find the angle the ladder makes with the ground, and the distance of its foot from the wall.

6. From the top of a hill the angles of depression of two successive milestones on a straight, level road leading to the hill are $5°$ and $15°$. Find the height of the hill. (Hint: Let $x =$ height of the hill, and $y =$ the distance from the bottom to the nearer milestone; then eliminate y from the two equations representing the cotangent of $15°$ and $5°$, respectively.)

7. From a point on the ground the angles of elevation of the bottom and the top of a tower on a building are $64° 17'$ and $68° 43'$. If the tower is 200 feet high, find the height of the building.

8. Two flag poles are known to be 60 and 40 feet high, respectively. A person moves about until he finds a position such that the tip of the nearer pole just hides that of the farther. At this point the angle of elevation of the top of the nearer pole is found

to be 35° 10′. Find the distance between the poles, and the distance from the observer to the nearer pole.

9. A tin roof rises $4\frac{1}{2}$ inches to the horizontal foot. Find the angle the roof makes with the horizontal.

10. From the top of a mountain, 2800 feet above a hut in the valley, the angle of depression of the hut is found to be 38°. Find the straight-line distance from the top of the mountain to the hut.

11. Find the height of a tree which casts a shadow of 80 feet when the angle of elevation of the sun is 40°.

12. From a ship's masthead 160 feet high, the angle of depression of a boat is 30°. Find the distance from the boat to the ship.

13. From the top of a cliff 150 feet high, the angles of depression of two boats at sea, each due south of the observer, are 32° and 20°, respectively. Find the distance between the boats.

14. A circle is inscribed in an equilateral triangle of perimeter 90 inches. Find the diameter of the circle.

15. Find the length of a chord which subtends a central angle of 64° in a circle whose radius is 10 feet.

16. From the foot of a post 30 feet high, the angle of elevation of the top of a steeple is 64°; and from the top of the post, the angle of depression of the base of the steeple is 50°. Find the height of the steeple.

17. From one end of a bridge the angle of depression of an object 200 feet downstream from the bridge and at the water line is 23°. From the same point the angle of depression of an object at the water line exactly under the opposite end of the bridge is 16°. Find the length of the bridge, and its height above the river.

18. From a point directly north of an inaccessible peak, the angle of elevation of the top is found to be 28°. From another point on the same level and directly west of the peak, the angle of elevation of the top is 40°. Find the height of the peak if the distance between the two points of observation is 6000 feet.

19. A tower stands on the bank of a river. The angle of elevation of the top of the tower from a point directly opposite on the other bank is 55°. From another point 100 feet beyond this point, the angle of elevation of the top of the tower is 28°. Find the height of the tower, and the width of the river.

20. A tree stands upon the same horizontal plane as a house which is 60 feet high. The angles of elevation and depression of the top and the base of the tree from the top of the house are 40° and 35°, respectively. Find the height of the tree.

21. The slope of a mountain 4000 feet high makes on one side an angle of 10° with the horizontal, and on the other side an angle of 12°. A man can walk up the steeper slope at the rate of 2 miles per hour, and up the easier slope at 3 miles per hour. Find the route by which he will reach the summit sooner. How many minutes sooner will he arrive?

Chapter IV

TRIGONOMETRIC FUNCTIONS OF ALL ANGLES

15. Positive and negative angles. In Sections 8 and 9, we discussed *acute* angles and defined the six trigonometric functions of acute angles. In this chapter we shall be concerned with angles of any magnitude, positive or negative, and the trigonometric functions of these angles. We first distinguish between positive and negative angles.

If an angle θ is generated by a *counter-clockwise* rotation, as in Figure 12, we call angle θ a *positive* angle.

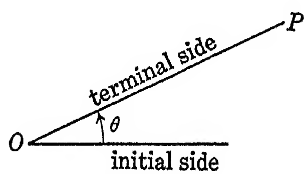

Figure 12.

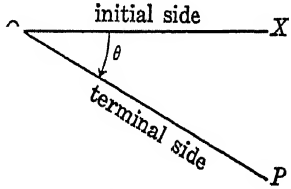

Figure 13.

On the other hand, if an angle θ is generated by a *clockwise* rotation, as in Figure 13, we call angle θ a *negative* angle.

By an angle of 390°, then, we mean an angle generated by OX revolving about O in a counter-clockwise direction, making one complete revolution of 360° and moving 30° in addition, as indicated in Figure 14.

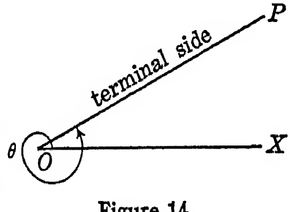

Figure 14.

There is a similar understanding for negative angles greater numerically than 360°, the rotation being clockwise.

16. Directed distances. Just as in the previous section we made a distinction between positive and negative angles, so now we make a distinction between positive and negative

distances. In general we call a line segment *positive* if it is generated by a point moving from left to right, as AB in Figure 15. We indicate such a line segment by AB.

Figure 15.

If the line segment is generated by a point moving from right to left, we call the line segment *negative*, and indicate it by BA. (Observe that AB means the segment generated from A to B, and that BA means from B to A.) Thus, in Figure 15,

$$BA = -AB.$$

But it is desirable often to distinguish between positive and negative distances when the direction is neither from left to right nor from right to left. In that case, we choose a given direction as positive, and consider the direction directly opposite as negative. Thus, in Figure 16, if the direction of the arrow indicates the positive direction, we have:

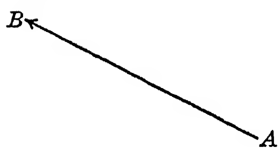

Figure 16.

$$BA = -AB,$$

or:

$$AB = -BA.$$

It is easy to prove that, if A, B, and C are three points on a line, then, regardless of the positions of A, B, and C,

$$AB + BC = AC.$$

17. Coördinates. Let $X'X$ and $Y'Y$ be two perpendicular lines intersecting at O. Let P be any point in the plane of these lines. For purposes of illustration, the point P is taken as in Figure 17. Drop AP perpendicular to OX, and draw OP. The length OA is denoted by x, and is called the x, or *abscissa*, of P. The length AP is denoted

by y, and is called the y, or *ordinate*, of P. The x and y taken together, thus (x, y), are called the *coördinates* of P. $X'X$ and $Y'Y$ are called the *axes* of the coördinates. O is called the *origin*, and r the *radius vector*, of P. If x is measured from left to right, we call it positive; if from right to left, negative. If y is measured upwards, it is positive; if downwards, negative. For convenience we assume r always to be positive.

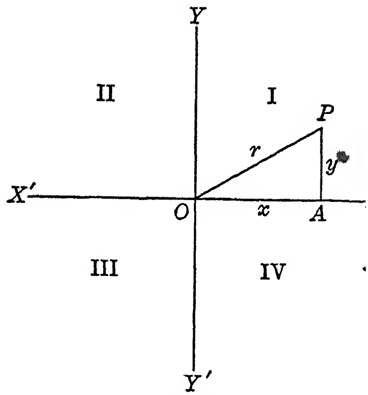

Figure 17.

18. Quadrants. The axes divide the plane into four parts, called *quadrants*, numbered as in Figure 17. It is quite apparent then that the following arrangement holds for the signs of the x and y of a point in the quadrants indicated.

P	x	y
I	+	+
II	−	+
III	−	−
IV	+	−

Thus the point $(-2, 3)$ lies in the second quadrant.

19. Trigonometric functions of all angles. We shall define the trigonometric functions of angles greater numerically than 90° by a process exactly like the process used in arriving at the definitions of the functions of acute angles. Take the vertex of a given angle at O (Figure 18), and the initial side along the x-axis. Take any point P on the

terminal side, and drop a perpendicular to the initial side, extended if necessary. This forms a right triangle, called the *triangle of reference*—which, it should be noted, however, does not contain angle θ unless it is acute.

We define the trigonometric functions of θ as follows:

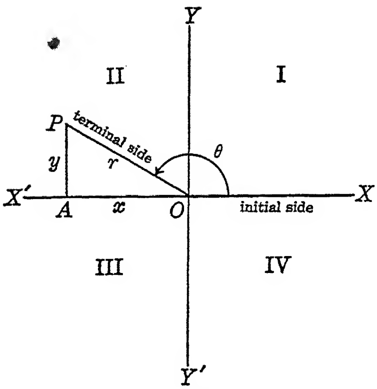

Figure 18.

$$\sin\theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{radius vector}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{radius vector}}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\csc\theta = \frac{r}{y} = \frac{\text{radius vector}}{\text{ordinate}}$$

$$\sec\theta = \frac{r}{x} = \frac{\text{radius vector}}{\text{abscissa}}$$

$$\cot\theta = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

Observe that, when θ is acute and therefore in the first quadrant, the above definitions reduce exactly to those given in Section 9.

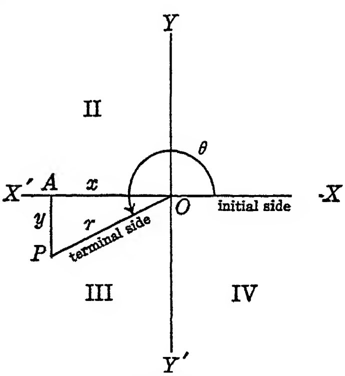

Figure 19*a*.

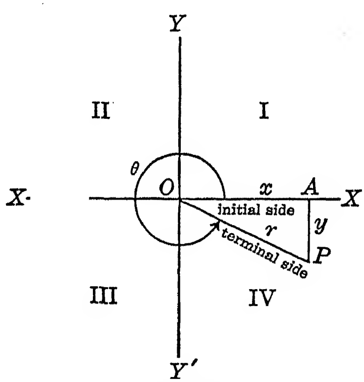

Figure 19*b*.

Figure 19*a* and Figure 19*b* illustrate the situation when θ is positive and in the third and fourth quadrants, respectively.

It is quite apparent that, except in the first quadrant where x, y, and r are all positive quantities, some of the above functions may at times be negative. The following table gives the arrangement of the signs of the functions of angle θ in the quadrants indicated; the signs are derived from the table in Section 18 and the definitions in Section 19.

θ	sin	cos	tan	csc	sec	cot
I	+	+	+	+	+	+
II	+	−	−	+	−	−
III	−	−	+	−	−	+
IV	−	+	−	−	+	−

Example

The sine of angle θ is $-\frac{3}{5}$, and the cosine of θ is negative. From Figure 20, find the other five functions.

Since $\sin\theta$ is negative and $\cos\theta$ is negative, θ lies in the third quadrant.

Then, since $\sin\theta = -\dfrac{3}{5}$

$$= \frac{-3}{5}$$

$$= \frac{y}{r},$$

we have: $x^2 = r^2 - y^2$

$$= 25 - 9$$

$$= 16.$$

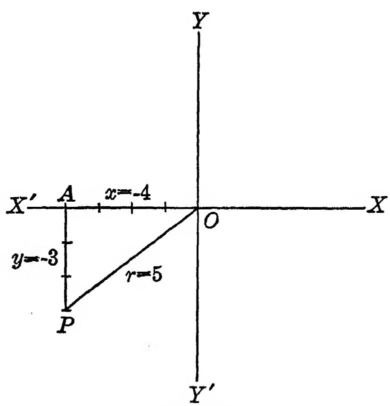

Figure 20.

Therefore: $x = \pm 4.$

But, since θ lies in the third quadrant, x must be negative.

Therefore: $x = -4.$

Hence we have the following:

$$\sin\theta = -\frac{3}{5}$$

$$\cos\theta = -\frac{4}{5}$$

$$\tan\theta = \frac{3}{4}$$

$$\csc\theta = -\frac{5}{3}$$

$$\sec\theta = -\frac{5}{4}$$

$$\cot\theta = \frac{4}{3}$$

Problems

1. Find in what quadrants the following angles lie: 346°; 214°; −120°; 750°; −600°; −423°; 542°; 6000°.

In the following problems, consider θ positive and between 0° and 360°.

2. Given $\cos\theta = -\frac{3}{5}$, $\tan\theta$ negative; find all functions of θ.
3. Given $\sec\theta = \frac{5}{4}$, $\sin\theta$ negative; find all functions of θ.
4. Given $\cot\theta = \frac{5}{12}$, $\csc\theta$ negative; find $\cos\theta$.
5. Given $\tan\theta = -\frac{1}{2}$, $\sin\theta$ positive; find $\cos\theta$.
6. Given $\sin\theta = \frac{1}{3}$, $\tan\theta$ positive; find $\cot\theta$.
7. Given $\tan\theta = \frac{2}{5}$, θ not in the first quadrant; find $\cos\theta$.
8. Given $\sin\theta = -\frac{1}{2}$, θ not in the fourth quadrant; find $\sec\theta$.
9. Given $\csc\theta = 4$; find $\cos\theta$.
10. Given $\tan\theta = 1$; find $\sin\theta$.

20. Functions of 0°, 90°, 180°, 270°, 360°. Consider an angle θ very close to 0° (Figure 21).

It is quite evident that, when $\theta = 0°$, point P coincides with A, and hence $x = r$, and $y = 0$. Therefore:

$$\sin 0° = \frac{y}{r} = \frac{0}{r} = 0$$

$$\cos 0° = \frac{x}{r} = \frac{r}{r} = 1$$

$$\tan 0° = \frac{y}{x} = \frac{0}{x} = 0$$

$$\csc 0° = \frac{r}{y} = \frac{r}{0} = \infty \quad \text{(Infinity*)}$$

$$\sec 0° = \frac{r}{x} = 1$$

$$\cot 0° = \frac{x}{y} = \frac{x}{0} = \infty$$

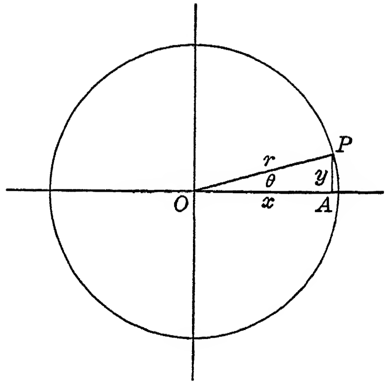

Figure 21.

Similarly, consider an angle θ very close to 90° (Figure 22). When θ = 90°, point P coincides with B, and we have $r = y$, and $x = 0$. Therefore:

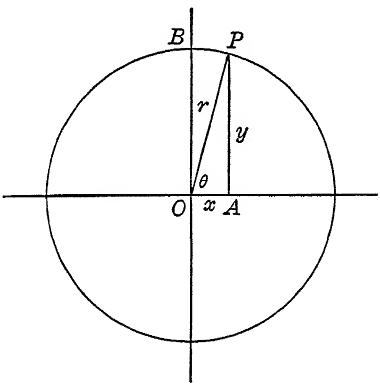

Figure 22.

$$\sin 90° = \frac{y}{r} = 1$$

$$\cos 90° = \frac{x}{r} = \frac{0}{r} = 0$$

$$\tan 90° = \frac{y}{x} = \frac{y}{0} = \infty$$

$$\csc 90° = \frac{r}{y} = 1$$

$$\sec 90° = \frac{r}{x} = \frac{r}{0} = \infty$$

$$\cot 90° = \frac{x}{y} = \frac{0}{y} = 0$$

Similarly, consider an angle θ very close to 180° (Figure 23). When θ = 180°, point P coincides with A. Hence we have $y = 0$, and $x = -r$; for x and r are equal numerically, but x is directed to the left and is therefore negative, and r is always positive. Therefore:

* By *infinity* we mean that, as θ approaches zero, y approaches zero, and $\frac{r}{y}$, that is, $\csc \theta$, increases without limit.

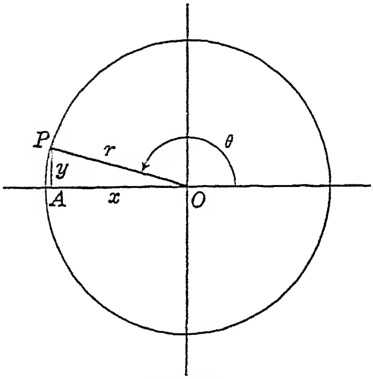

Figure 23.

$$\sin 180° = \frac{y}{r} = \frac{0}{r} = 0$$

$$\cos 180° = \frac{x}{r} = \frac{-r}{r} = -1$$

$$\tan 180° = \frac{y}{x} = \frac{0}{x} = 0$$

$$\csc 180° = \frac{r}{y} = \frac{r}{0} = \infty$$

$$\sec 180° = \frac{r}{x} = \frac{r}{-r} = -1$$

$$\cot 180° = \frac{x}{y} = \frac{x}{0} = \infty$$

Similar results obtain for the functions of 270°, which are tabulated below; and it may readily be seen that the functions of 360° are the same as the corresponding functions of 0°.

θ	sin	cos	tan	csc	sec	cot
0°	0	1	0	$\pm\infty$	1	$\pm\infty$
90°	1	0	$\pm\infty$	1	$\pm\infty$	0
180°	0	-1	0	$\pm\infty$	-1	$\pm\infty$
270°	-1	0	$\pm\infty$	-1	$\pm\infty$	0
360°	0	1	0	$\pm\infty$	1	$\pm\infty$

21. Functions of θ as θ varies from 0° to 360°. Let us, by means of the above table, study the development of the various functions of an angle as the angle varies from 0° to 360°.

First, consider $\sin\theta$. Sin θ starts at 0 when θ is 0°. Then, as θ increases to 90°, $\sin\theta$ increases to $+1$, taking all values between 0 and 1. As θ increases from 90° to

180°, sin θ decreases from 1 to 0. As θ increases from 180° to 270°, sin θ decreases from 0 to -1. Finally, as θ increases from 270° to 360°, sin θ increases from -1 to 0. This process repeats itself as θ continues to increase.

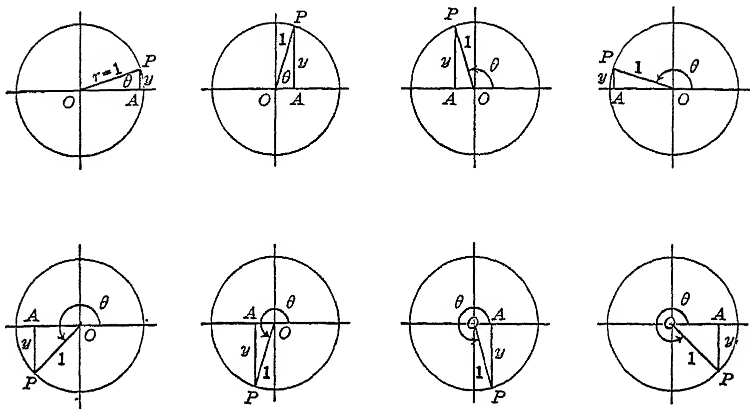

Figure 24.

The process may be illustrated geometrically. Let us consider a circle of radius 1, called a *unit circle* (Figure 24). Then

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y.$$

Sin θ may be represented by the line AP. It is quite apparent from Figure 24 that the development of sin θ as θ varies from 0° to 360° is as stated above.

Cos θ behaves similarly. When θ is 0°, cos θ is 1. As θ increases from 0° to 90°, cos θ decreases from 1 to 0. As θ increases from 90° to 180°, cos θ decreases from 0 to -1. As θ increases from 180° to 270°, cos θ increases from -1 to 0. As θ increases from 270° to 360°, cos θ increases from 0 to 1.

When θ is 0°, tan θ is 0. As θ increases from 0° to 90°, tan θ increases from 0 to ∞. Since the tangent function is negative in the second quadrant and very large numerically for angles slightly larger than 90°, and since it increases

without limit if we approach $90°$ through such angles, we say $\tan 90°$ equals $\pm\infty$. Hence, as θ increases from $90°$ to $180°$, $\tan\theta$ increases from $-\infty$ to 0. As θ increases from $180°$ to $270°$, $\tan\theta$ increases from 0 to $+\infty$. As θ increases from $270°$ to $360°$, $\tan\theta$ increases from $-\infty$ to 0.

When θ is $0°$, $\csc\theta$ is ∞. As θ increases from $0°$ to $90°$, $\csc\theta$ decreases from ∞ to 1. As θ increases from $90°$ to $180°$, $\csc\theta$ increases from 1 to ∞. As θ increases from $180°$ to $270°$, $\csc\theta$ increases from $-\infty$ to -1. As θ increases from $270°$ to $360°$, $\csc\theta$ decreases from -1 to $-\infty$.

When θ is $0°$, $\sec\theta$ is 1. As θ increases from $0°$ to $90°$, $\sec\theta$ increases from 1 to ∞. As θ increases from $90°$ to $180°$, $\sec\theta$ increases from $-\infty$ to -1. As θ increases from $180°$ to $270°$, $\sec\theta$ decreases from -1 to $-\infty$. As θ increases from $270°$ to $360°$, $\sec\theta$ decreases from ∞ to 1.

When θ is $0°$, $\cot\theta$ is ∞. As θ increases from $0°$ to $90°$, $\cot\theta$ decreases from ∞ to 0. As θ increases from $90°$ to $180°$, $\cot\theta$ decreases from 0 to $-\infty$. As θ increases from $180°$ to $270°$, $\cot\theta$ decreases from $+\infty$ to 0. As θ increases from $270°$ to $360°$, $\cot\theta$ decreases from 0 to $-\infty$.

The statement $\tan 90° = \infty$ may be illustrated geometrically by means of a unit circle (Figure 25a).

$$\tan\theta = \frac{AP}{1} = AP$$

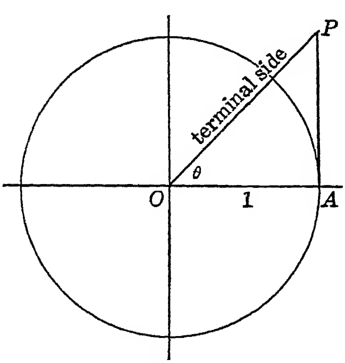

Figure 25*a*.

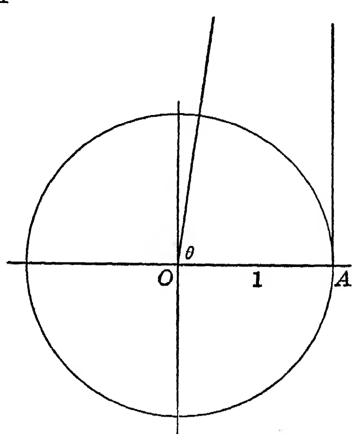

Figure 25*b*.

P is determined as the intersection point of a tangent to the circle at A and the terminal side of angle θ. It is quite apparent that, as θ approaches 90°, the terminal side of θ approaches parallelism with the tangent at A, and P recedes indefinitely (Figure 25*b*).

From the above discussion we see that the sine and cosine of an angle are always between -1 and $+1$, or equal to -1 or $+1$; that the tangent and cotangent may take any values; that the secant and cosecant are never between -1 and $+1$, but may take all other values, including -1 and $+1$.

22. Functions of (180° ± θ) and (360° ± θ). In the previous section, we observed that, as θ varied from 0° to 90° to 180°, $\sin\theta$ varied from 0 to 1 to 0. It is quite evident, then, if we consider a number between 0 and 1—say $\frac{3}{5}$—that there is an angle in the first quadrant whose sine is $\frac{3}{5}$ and that there is, also, an angle in the second quadrant whose sine is $\frac{3}{5}$. What is the relation between these two angles? Our answer is: they must be suplementary; that is, the sum of the two angles must equal 180°. We shall prove a general theorem regarding such angles.

Theorem:

$$\sin(180° - \theta) = \sin\theta.$$

Proof

Given the right triangle ABO (see Figure 26), with $\angle XOA = (180° - \theta)$. Construct $\angle COX$ equal to θ. Take $OP = r = r'$. Drop a perpendicular from P to OX at M. Then the right triangles MOP and BOA are congruent. Letter as in Figure 26.

Hence we have:

$$r = r',$$
$$y = y',$$
$$x = -x'.$$

The sine of $\angle XOA = (180° - \theta)$ is $\frac{y'}{r'}$ (see Figure 26). This equals $\frac{y}{r}$, since $y' = y$ and $r' = r$. But $\frac{y}{r}$ (see Figure 26) equals sine θ. Or, restated,

$$\sin(180° - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin\theta.$$

$$\therefore \sin(180° - \theta) = \sin\theta.$$

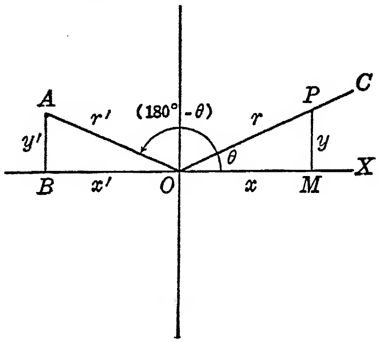

Figure 26.

Similarly,

$$\cos(180° - \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\therefore \cos(180° - \theta) = -\cos\theta.$$

Similarly,

$$\tan(180° - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\frac{y}{x} = -\tan\theta.$$

$$\therefore \tan(180° - \theta) = -\tan\theta.$$

In like fashion,

$$\csc(180° - \theta) = \csc\theta,$$
$$\sec(180° - \theta) = -\sec\theta,$$
$$\cot(180° - \theta) = -\cot\theta.$$

Example

Find: sin 150°; cot 135°.

$$\sin 150° = \sin(180° - 30°) = \sin 30° = \tfrac{1}{2}.$$
$$\cot 135° = \cot(180° - 45°) = -\cot 45° = -1.$$

Similar results obtain for the functions of $(180° + \theta)$, as indicated in the following text:

Proof

In Figure 27, we have:

$$r = r',$$
$$x = -x',$$
$$y = -y'.$$

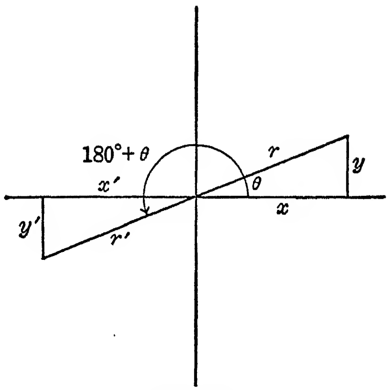

Figure 27.

Hence we have the following:

$$\sin\,(180° + \theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$$

$$\cos\,(180° + \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta$$

$$\tan\,(180° + \theta) \quad y' \quad \frac{-y}{-x} = \frac{y}{x} = \tan\theta$$

$$\csc\,(180° + \theta) = -\csc\theta$$
$$\sec\,(180° + \theta) = -\sec\theta$$
$$\cot\,(180° + \theta) = \cot\theta$$

Similarly, we have these results:

$$\sin\,(360° - \theta) = -\sin\theta$$
$$\cos\,(360° - \theta) = \cos\theta$$
$$\tan\,(360° - \theta) = -\tan\theta$$

$$\csc(360° - \theta) = -\csc\theta$$
$$\sec(360° - \theta) = \sec\theta$$
$$\cot(360° - \theta) = -\cot\theta$$

From the above discussions we may state the following theorem.

Theorem. Any function of $(180° \pm \theta)$ *or* $(360° \pm \theta)$*—in fact, of* $(n180° \pm \theta)$*, where* n *is a positive integer—is equal to the same function of* θ*, with the sign depending upon the quadrant in which the angle lies.*

Thus:

$$\cos 240° = \cos(180° + 60°)$$
$$= -\cos 60°$$

(since 240° is in the third quadrant and cosine is negative there)

$$= -\frac{1}{2}.$$

In the above proofs, θ was taken as acute. The theorem holds, however, regardless of the size of θ, as does also the information about to be obtained regarding a negative angle.

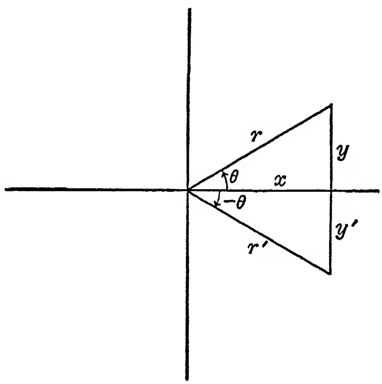

Figure 28.

23. Functions of $(-\theta)$. What relations hold between the functions of a negative angle and the functions of the corresponding positive angle? Consider Figure 28.

$$r \quad r'$$
$$x \quad x$$
$$y = -y'$$
$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$$
$$\cos(-\theta) \qquad \frac{x}{r} \qquad +\cos\theta$$
$$\tan(-\theta) = \frac{y'}{x} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\sec(-\theta) = \sec\theta$$
$$\cot(-\theta) = -\cot\theta$$

Problems

Example 1

Find: $\sin - 300°$.

$$\begin{aligned}\sin - 300° &= -\sin 300°\\ &= -\sin(360° - 60°)\\ &= -(-\sin 60°)\\ &= \sin 60°\\ &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Example 2

Find all angles between 0° and 360° which satisfy the equation: $2\sin^2\theta - 3\sin\theta + 1 = 0$.

$$2\sin^2\theta - 3\sin\theta + 1 = (2\sin\theta - 1)(\sin\theta - 1) = 0.$$

Hence: $\sin\theta = \frac{1}{2}$,

or: $\sin\theta = 1$.

If $\sin\theta = \frac{1}{2}$,

then $\theta = 30°$ or $150°$.

If $\sin\theta = 1$,

then $\theta = 90°$.

Hence: $\theta = 30°, 150°$, or $90°$.

1. In Problems (*a*) to (*l*), the student is requested not to use the tables. Find:

(a) $\sin 210°$
(b) $\cot 315°$.
(c) $\tan 240°$.
(d) $\csc 120°$.
(e) $\sec 225°$.
(f) $\cos 120°$.
(g) $\tan -330°$.
(h) $\sin -150°$.
(i) $\cos -135°$.
(j) $\cot -300°$.
(k) $\csc -120°$.
(l) $\sec -330°$.

2. Solve the following equations to find, for the unknown letters, all values between 0° and 360°.

(a) $\cos \theta + \dfrac{1}{\cos \theta} = \dfrac{5}{2}$.

(b) $\tan^2 \theta + \dfrac{3}{\tan^2 \theta} - 4 = 0$.

(c) $\tan \theta + \dfrac{1}{\tan \theta} = 4$.

(d) $\sin x + \dfrac{1}{\sin x} = 3$.

(e) $\tan x + \sqrt{3} = 0$.
(f) $3 \sin^2 x - 5 \sin x + 2 = 0$.
(g) $\sin 2x = 0$.
(h) $2 \sin^2 x + \sin x - 1 = 0$.
(i) $(\tan^2 \theta - 3)(\csc \theta - 2) = 0$.
(j) $\sin^2 x = \sin x$.
(k) $2 \sin^2 x = \sqrt{3} \sin x$.
(l) $\sin^2 \theta - 5 \sin \theta + 6 = 0$.
(m) $6 \cos^2 x - 5 \cos x + 1 = 0$.

CHAPTER V

THE OBLIQUE TRIANGLE

24. Law of sines. In Chapter III we considered the solution of the right triangle. In this chapter we shall be concerned with the solution of the oblique triangle, for which we use two important laws giving relations between the sides and the angles of such a triangle. We proceed to the derivation of the first of these laws, called the *law of sines:*

Law of Sines. *The sides of a triangle are proportional to the sines of the angles opposite.*

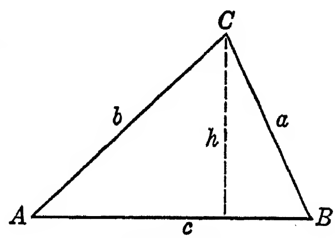

Figure 29.

Consider the triangle ABC in Figure 29. We wish to prove:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Proof

From C, drop h perpendicular to AB.

Then $$\sin A = \frac{h}{b},$$

and $$\sin B = \frac{h}{a}.$$

Dividing, $$\frac{\sin A}{\sin B} = \frac{h/b}{h/a} = \frac{a}{b},$$

or: $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, it may be shown that

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

or

$$\frac{\quad}{\sin B} = \frac{\quad}{\sin C}$$

Hence:

$$\frac{\quad}{\sin A} = \frac{\quad}{\sin B} = \frac{\quad}{\sin C}$$

In Figure 29, all the angles were taken as acute. The law holds, however, if an angle is obtuse, as A in Figure 30.

Proof

Drop h perpendicular to c, extended to M.

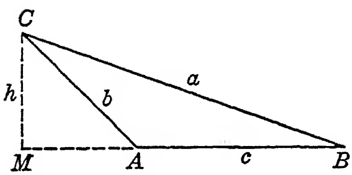

Figure 30.

Then $\sin B = \dfrac{h}{a}$

and $\sin A = \dfrac{h}{b}$

by the definition of the sine of an angle in the second quadrant. The rest of the proof is similar to the preceding proof.

25. Applications of the law of sines. By using the law of sines, we are enabled to solve a triangle if we are given any two angles and a side, as in the following:

Example

Given $A = 52° \ 13'$, $B = 73° \ 24'$, $c = 6293$. Solve the triangle.

$$\begin{aligned} C &= 180° - (A + B) \\ &= 179° \, 60' - 125° \, 37' \\ \therefore C &= 54° \, 23' \end{aligned}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{c \sin A}{\sin C}$$

$$
\begin{aligned}
\log a &= \log c + \log \sin A - \log \sin C \\
\log 6293 &= 3.7989 \\
\log \sin 52^\circ\, 13' &= \underline{9.8978 - 10} \\
&\ 13.6967 - 10 \\
\log \sin 54^\circ\, 23' &= \underline{9.9101 - 10} \\
\therefore \log a &= 3.7866 \\
\therefore a &= 6118
\end{aligned}
$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$
\begin{aligned}
\log b &= \log c + \log \sin B - \log \sin C \\
\log 6293 &= 3.7989 \\
\log \sin 73^\circ\, 24' &= \underline{9.9815 - 10} \\
&\ 13.7804 - 10 \\
\log \sin 54^\circ\, 23' &= \underline{9.9101 - 10} \\
\therefore \log \cancel{c}\, b &= 3.8703 \\
\therefore b\, \cancel{c} &= 7418
\end{aligned}
$$

Hence:

$$
\begin{aligned}
C &= 54^\circ\, 23', \\
a &= 6118, \\
b\ \cancel{c} &= 7418.
\end{aligned}
$$

Problems

Solve the following triangles:

1. $a = 26.32$, $A = 46^\circ\, 52'$, $B = 64^\circ\, 43'$.
2. $a = 406.2$, $B = 19^\circ\, 36'$, $C = 80^\circ\, 52'$.
3. $b = 6601$, $A = 50^\circ\, 32'$, $C = 100^\circ$.
4. $c = 32.04$, $A = 25^\circ\, 42'$, $B = 40^\circ\, 19'$.
5. $c = 530$, $A = 46^\circ\, 10'$, $B = 63^\circ\, 50'$.

26. Ambiguous case. The other type of triangle handled by the law of sines is that for which there are given two sides and one angle opposite one of the given sides. This case presents slightly more difficulty, however, because, with the above material, there may be no triangle possible, there may be one and one only, or there may be two triangles possible both of which contain the given material.

This case is therefore known as the *ambiguous case.* Let us consider a concrete illustration:

Example

Given $A = 30°$, $a = 75$, $b = 100$.

In Figure 31, we construct on AX at A an angle of 30°, with sides AX and AM. On AM we lay off 100 units (b) from A, say AC. With C as center and ($a = 75$) as radius, we swing an arc. The number of solutions depends on whether or not the arc cuts AX and, if so, where. In this particular case, the arc cuts AX at two points B' and B, both to the right of A. Since both triangles, ACB and ACB', contain the given material, there are consequently two solutions.

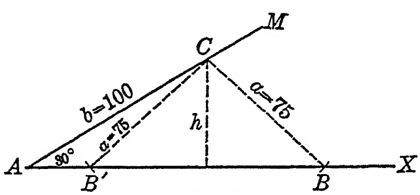

Figure 31.

Obviously there will always be two solutions when the length of a is numerically less than b and greater than the perpendicular h, dropped from C. Also, there will never be a solution if a is less than h. There will be but one solution, a right triangle, if a equals h; and there will be but one solution, an isosceles triangle, if a is greater than h and equal to b. Finally, there will be but one solution if a is greater than h and greater than b; for, although the arc will cut AX at two points, one will be to the left of A and the triangle thus formed will not include A.

The matter of finding h is very simple:

Since $$\frac{h}{b} = \sin A,$$

$$h = b \sin A.$$

In Figure 31, $$h = 100 \cdot \tfrac{1}{2}$$
$$= 50.$$

The above proof and explanation obtain when A is acute. If A is obtuse, there will be one and only one solution provided a is greater than b, and then only.

We may summarize our findings as follows:

Case I. Given A, a, and b, with A acute. There will be two solutions if $b \sin A$ is less than a, and a is less than b. There will be one solution if $b \sin A$ equals a, or if a equals b, or if a is greater than b. There will be no solution if a is less than $b \sin A$.

Case II. Given A, a, and b, with A obtuse. There will be one solution if a is greater than b.

In solving a triangle, the student should realize that the only possible chance for there being two solutions is in the case when A is acute and a is less than b. If such is the case, the student should then find the relation between a and $b \sin A$, and proceed accordingly.

Example

Solve the triangle; given $a = 800$, $b = 1200$, $A = 34°$. (See Figure 32. This is obviously a chance for two solutions.)

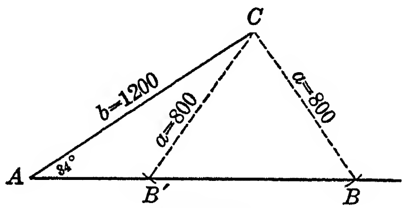

Figure 32.

Using logs,

$$\begin{aligned} \log b &= \log 1200 = 3.0792 \\ \log \sin A &= \log \sin 34° = 9.7476 - 10 \\ \log b \sin A &= 12.8268 - 10 \\ \log a &= \log 800 = 2.9031 \end{aligned}$$

(Since $\log a$ is greater than $\log b \sin A$, there will be two solutions.)

We first solve triangle ABC in Figure 32.

$$\frac{\sin B}{b} \quad \frac{\sin A}{a}$$

$$\sin B \quad \frac{b \sin A}{a}$$

$$\log \sin B = \log b + \log \sin A - \log a = 9.9237 - 10$$
$$\therefore B = 57^\circ\ 1'$$
$$C = 180^\circ - (A + B) = 180^\circ \quad 91^\circ\ 1'$$
$$\therefore C = 88^\circ\ 59'$$

$$\sin C \qquad \sin A$$

$$\log c = \log a + \log \sin C - \log \sin A$$

$$\begin{aligned}
\log 800 &= 2.9031 \\
\log \sin 88^\circ\ 59' &= \underline{9.9999 - 10} \\
&= 12.9030 - 10 \\
\log \sin 34^\circ &= \underline{9.7476 - 10} \\
\log c &= 3.1554 \\
\therefore c &= 1430
\end{aligned}$$

We next solve triangle $AB'C$, in Figure 32. Let $AB' = c'$, and $\angle ACB' = C'$.

$$B = 57^\circ\ 1'$$
$$\therefore B' = 180^\circ - 57^\circ\ 1' = 122^\circ\ 59'$$
$$\therefore C' = \angle ACB' = 180^\circ - (122^\circ\ 59' + 34^\circ) = 23^\circ\ 1'$$
$$\frac{c'}{\sin C'} = \frac{a}{\sin A}$$
$$\log c' = \log a + \log \sin C' - \log \sin A$$

$$\begin{aligned}
\log 800 &= 2.9031 \\
\log \sin 23^\circ\ 1' &= \underline{9.5922 - 10} \\
&= 12.4953 - 10 \\
\log \sin 34^\circ &= \underline{9.7476 - 10} \\
\log c' &= 2.7477 \\
\therefore c' &= 559.4
\end{aligned}$$

Hence, for $\triangle ABC$: $B = 57^\circ\ 1'$,
$C = 88^\circ\ 59'$,
$c = 1430$;

and, for $\triangle AB'C$:

$$B' = 122° \, 59',$$
$$C' = 23° \, 1',$$
$$c' = 559.4.$$

Problems

1. Find the number of solutions; given:

(a) $A = 30°$, $b = 200$, $a = 101$.
(b) $A = 30°$, $b = 400$, $a = 100$.
(c) $A = 30°$, $b = 600$, $a = 300$.
(d) $A = 30°$, $b = 500$, $a = 500$.
(e) $A = 30°$, $b = 500$, $a = 600$.

2. Find the number of solutions; given:

(a) $A = 150°$, $b = 200$, $a = 150$.
(b) $A = 150°$, $b = 200$, $a = 300$.
(c) $A = 150°$, $b = 200$, $a = 200$.
(d) $B = 150°$, $b = 200$, $a = 300$.
(e) $B = 150°$, $c = 400$, $b = 300$.

3. Solve the following triangles:

(a) $A = 59° \, 26'$, $a = 7072$, $b = 7836$.
(b) $A = 140° \, 26'$, $a = 40.34$, $b = 30.29$
(c) $A = 32° \, 14'$, $a = 464.7$, $b = 600.8$.
(d) $B = 47° \, 46'$, $b = 3247$, $a = 3015$.
(e) $C = 62° \, 34'$, $c = 375$, $a = 400$.
(f) $A = 65° \, 53'$, $a = 20.43$, $b = 30.32$.

27. Law of cosines, and applications. There are two other types of triangles to be considered; that is, triangles with three sides given, or triangles with two sides and the included angle given. These two types are handled by the *law of cosines.* We proceed to its derivation.

Law of Cosines. *The square of any side of a triangle equals the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of the included angle.*

Given the triangle ABC, in Figure 33. We wish to prove:

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}}$$

Let us prove the first relation:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

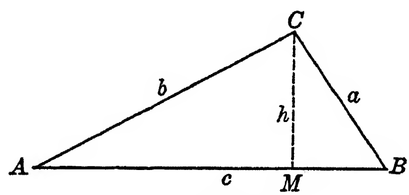

Figure 33.

Proof

Drop a perpendicular h from C to AB at M.

Then $$h^2 = b^2 - \overline{AM}^2,$$
and $$h^2 = a^2 - \overline{MB}^2.$$
Equating, we have:

$$a^2 - \overline{MB}^2 = b^2 - \overline{AM}^2,$$

or: $$a^2 = b^2 + \overline{MB}^2 - \overline{AM}^2$$

Then, since $$MB = c - AM$$
and $$\overline{MB}^2 = c^2 - 2c(AM) + \overline{AM}^2,$$

substituting, we have:

$$a^2 = b^2 + c^2 - 2c(AM) + \overline{AM}^2 - \overline{AM}^2.$$

Or: $$a^2 = b^2 + c^2 - 2c(AM).$$

But $$\frac{AM}{b} = \cos A,$$

or $$AM = b(\cos A).$$

Therefore: $$a^2 - b^2 + c^2 - 2bc \cos A.$$

In Figure 33, angle A was taken as acute. We shall now show that the law holds when A is obtuse, as in Figure 34.

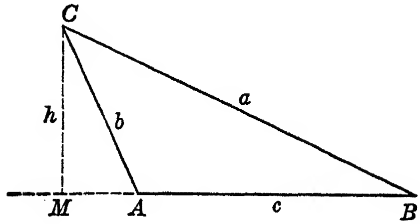

Figure 34.

Proof

Drop a perpendicular h (equal to CM) from C to BA, extended.

As before, $\quad h^2 = b^2 - \overline{MA}^2,$

and $\quad h^2 = a^2 - \overline{MB}^2.$

Then $\quad a^2 - \overline{MB}^2 = b^2 - \overline{MA}^2.$

Hence:

$$
\begin{aligned}
a^2 &= b^2 + \overline{MB}^2 - \overline{MA}^2 \\
&= b^2 + (MA + c)^2 - \overline{MA}^2 \\
&= b^2 + c^2 + 2c(MA) + \overline{MA}^2 - \overline{MA}^2 \\
&= b^2 + c^2 + 2c(MA).
\end{aligned}
$$

But $\quad \cos A \quad \dfrac{AM}{b}$

from the definition of the cosine of an angle in the second quadrant. However, since

$$AM = -MA,$$

then

$$\cos A = -\frac{MA}{b}.$$

Therefore: $\quad MA \quad -b \cos A.$

Hence, substituting, $\quad a^2 \quad b^2 + c^2 - 2bc \cos A.$

The other two relations may be derived in similar fashion We may apply the law of cosines as in the two examples below.

Example 1

Given $a = 4$, $b = 5$, $c = 6$; solve the triangle.

Since $\quad a^2 = b^2 + c^2 - 2bc \cos A,$

then $$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

or $$\frac{25 + 36 - 16}{60} = \frac{45}{60} = \frac{3}{4} = .7500.$$

$$\therefore A = 41° \, 25'.$$

Similarly, $$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

or $$\frac{16 + 36 - 25}{48} = \frac{27}{48} = \frac{9}{16}.$$

$$\begin{aligned} \log \cos B &= \log 9 - \log 16 \\ \log 9 &= 10.9542 - 10 \\ \log 16 &= \underline{\ \ 1.2041 \qquad\quad} \\ \log \cos B &= \ \ 9.7501 - 10 \\ \therefore B &= 55° \, 46'. \end{aligned}$$

Similarly, $$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

or $$\frac{16 + 25 - 36}{40} = \frac{5}{40} = \frac{1}{8} = .1250.$$

$$\therefore C = 82° \, 49'.$$

As a check: $A + B + C = 180°.$

Of course, as soon as we had found A, we could have used the law of sines to find B and subtracted $(A + B)$ from 180° to find C. But with convenient numbers, as in this example, it is fully as easy to proceed as above.

Example 2

Solve the triangle; given $a = 20$, $b = 25$, $C = 60°$.

Solving, $$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 400 + 625 - 2 \cdot 500 \cdot \tfrac{1}{2} \\ &= 1025 - 500 \\ &= 525. \\ \therefore c &= 22.91. \end{aligned}$$

We can now find A and B either by the law of cosines or by the law of sines.

It is apparent from the above examples that the law of cosines is not particularly well adapted for the use of logarithms. There are formulas which are better fitted for logarithmic use, but it is the author's feeling that an intelligent use of the tables of squares and square roots combined with the law of cosines is fully as easy and does not involve remembering a set of formulas and their derivations, which are not particularly essential. We shall illustrate in the following:

Example 3

Solve the triangle; given $a = 20.63$, $b = 34.21$, $c = 40.17$.

We have: $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

By the table of squares,
$$a^2 = 425.6,$$
$$b^2 = 1171,$$
$$c^2 = 1614.$$

(The interpolation is exactly the same as in logarithms.)

Hence: $$\cos A = \frac{2359.4}{2 \times 40.17 \times 34.21}$$

Now we can use logarithms:

$$\log \cos A = \log 2359 - \log 2 - \log 40.17 - \log 34.21$$

$$\begin{aligned}
\log 2 &= .3010 \\
\log 40.17 &= 1.6039 \\
\log 34.21 &= \underline{1.5341} \\
\log \text{denominator} &= 3.4390 \\
\log 2359 &= 13.3727 - 10 \\
\log \text{denominator} &= \underline{3.4390 } \\
\log \cos A &= 9.9337 - 10 \\
\therefore A &= 30° 51'.
\end{aligned}$$

We may proceed similarly to find B and C; or we may use the law of sines. The latter procedure would probably be easier in this example.

Problems

1. Solve the following triangles:

(a) $a = 5, b = 7, c = 10$.
(b) $a = 4, b = 6, C = 60°$.
(c) $a = 5, b = 8, C = 120°$.
(d) $a = 12, b = 20, c = 25$.
(e) $a = 19.62, b = 28.43, c = 22.06$.
(f) $a = 14.72, c = 25.39, B = 22° 17'$.
(g) $a = 2032, b = 2491, c = 3824$.
(h) $a = 1.32, b = 2.63, c = 1.91$.
(i) $b = 2.04, c = 3.96, A = 135° 27'$.
(j) $a = 423.1, c = 500.2, B = 47° 43'$.

2. Show that the area of a triangle may be written as one-half the product of any two sides and the sine of the included angle.

3. Show that the radius R of the circle circumscribed about a triangle ABC is given by

$$2R = \qquad \sin A \qquad \sin B \qquad \sin C$$

4. Show that the area of any quadrilateral equals one-half the product of the diagonals and the sine of one of the included angles.

5. In a parallelogram, the sides are 6 and 15, and the smaller vertex angles are 50°. Find the lengths of the diagonals.

6. A and B are points 300 feet apart on the edge of a river, and C is a point on the opposite side. If the angles CAB and CBA are 70° and 63°, respectively, find the width of the river.

7. From a mountain top 3000 feet above sea level, two ships are observed, one north and the other northeast. The angles of depression are 11° and 15°. Find the distance between the ships.

8. A tower stands on one bank of a river. From the opposite bank, the angle of elevation of the tower is 61°; and from a point 45 feet farther inland, the angle of elevation is 51°. Find the width of the river.

9. A cliff 400 feet high is seen due south of a boat. The top of the cliff is observed to be at an elevation of 30°. After the boat travels a certain distance southwest, the angle of elevation is found to be 34°. Find how far the boat has gone from the first point of observation.

10. A vertical tower makes an angle of 120° with the inclined plane on which it stands. At a distance of 80 feet from the base of the tower—measured down the plane—the angle subtended by the tower is 22°. Find the height of the tower.

11. Two persons stand facing each other on opposite sides of a pool. The eye of one is 4 feet 8 inches above the water, and that of the other, 5 feet 4 inches. Each observes that the angle of depression of the reflection in the pool of the eye of the other is 50°. Find the width of the pool.

12. A flag pole stands on a hill which is inclined 17° to the horizontal. From a point 200 feet down the hill, the angle of elevation of the top of the pole is 25°. Find the height of the pole.

13. A tower 100 feet high stands on a cliff beside a river. From a point on the other side of the river and directly across from the tower, the angle of elevation of the top of the tower is 35°, and that of the base of the tower is 24°. Find the width of the river.

14. A ladder leaning against a house makes an angle of 40° with the horizontal. When its foot is moved 10 feet nearer the house, the ladder makes an angle of 75° with the horizontal. Find the length of the ladder.

15. Two forces—one of 10 pounds and the other of 7 pounds—make an angle of 24° 42′. Find the intensity and the direction of their resultant.

16. Two men a mile apart on a horizontal road observe a balloon directly over the road. The angles of elevation of the balloon are estimated by the men to be 62° and 76°. Find the height of the balloon above the road.

17. Two points A and B are separated by a swamp. To find the length of AB, a convenient point C is taken outside the swamp; and AC, BC, and angle ACB are found as follows: $AC = 932$ feet, $BC = 1400$ feet, and $ACB = 120°$. Find AB.

18. An observer is on a cliff 200 feet above the surface of the sea. A gull is hovering above him, and its reflection in the sea can be seen by the observer. He estimates the angle of eleva-

tion of the gull to be 30°, and the angle of depression of its reflection in the water to be 55°. Find the height of the gull above the sea.

19. An electric sign 40 feet high is put on the top of a building. From a point on the ground, the angles of elevation of the top and the bottom of the sign are 40° and 32°. Find the height of the building.

20. A cliff with a lighthouse on its edge is observed from a boat; the angle of elevation of the top of the lighthouse is 25°. After the boat travels 900 feet directly toward the lighthouse, the angles of elevation of the top and the base are found to be 50° and 40°, respectively. Find the height of the lighthouse.

21. Two trains start at the same time from the same station upon straight tracks making an angle of 60°. If one train runs 45 miles an hour and the other 55 miles an hour, find how far apart they are at the end of 2 hours.

22. From the top of a lighthouse, the angle of depression of a buoy boat at sea is 50°; and the angle of depression of a second buoy—300 feet farther out to sea but in a straight line with the first buoy—from the top of the lighthouse is 28°. Find the height of the lighthouse.

23. A flag pole 50 feet high stands on the top of a tower. From an observer's position near the base of the tower, the angles of elevation of the top and the bottom of the pole are 36° and 20°, respectively. Find the distance from the observer's position to the base of the tower.

24. A lighthouse sighted from a ship bears 70° east of north. After the ship has sailed 6 miles due south, the lighthouse bears 40° east of north. Find the distance of the ship from the lighthouse at each time of observation.

25. Two trees on a horizontal plane are 60 feet apart. A person standing at the base of one tree observes the angle of elevation of the top of the second. Then, standing at the base of the second tree, he observes that the angle of elevation of one tree is double that of the other. When the observer stands half-way between the trees, the angles of elevation are complementary. Find the height of each tree.

26. Two points are in a line, horizontally, with the base of a tower. Let α be the angle of elevation of the top of the tower from the nearer point, and β the angle of elevation from the far-

ther point. Show that, if d represents the distance between the points, the height of the tower is

$$\frac{d \sin \alpha \sin \beta}{\sin (\alpha - \beta)}.$$

27. A man on a cliff, at a height of 1320 feet, looks out across the ocean. (The radius of the earth is assumed to be 4000 miles.) Find the distance from the man to the horizon seen by him.

28. Find how high an observer must be above the surface of the ocean to see an object 30 miles distant on the surface.

29. From the top of a building at a distance d from a tower, the angle of elevation of the top of the tower is α, and the angle of depression of the base is β. Show that the height of the tower is

$$\frac{d \sin (\alpha + \beta)}{\cos \alpha \cos \beta}.$$

30. If r is the radius of the earth, h the height of an observer above sea level, and α the angle of depression of the observer's horizon, show that

$$\tan \alpha = \frac{\sqrt{2rh + h^2}}{\quad}.$$

31. A balloon is overhead. An observer, due north, estimates the angle of elevation to be α. Another observer, at a distance d due west from the first observer, figures his angle of elevation to be β. Show that the height of the balloon above the observers is

$$\frac{d \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}.$$

Chapter VI

TRIGONOMETRIC RELATIONS

28. Fundamental identities. This chapter is concerned with relations of the trigonometric functions of angles of various size and formation. In the present section we shall derive the so-called *fundamental identities.* Although, in Figure 35, the angle under consideration is acute, any angle might have been used.

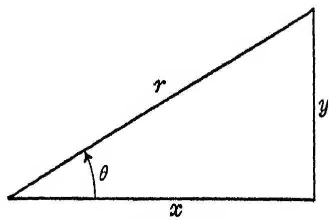

Figure 35.

The following relations are immediate consequences of the definitions of the six trigonometric functions of an angle:

$$(1)\quad \csc\theta = \frac{1}{\sin\theta}$$

$$(2)\quad \sec\theta = \frac{1}{\cos\theta}$$

$$(3)\quad \cot\theta = \frac{1}{\tan\theta}$$

$$(4)\quad \frac{1}{\csc\theta} = \sin\theta$$

$$(5)\quad \frac{1}{\sec\theta} = \cos\theta$$

$$(6)\quad \frac{1}{\cot\theta} = \tan\theta$$

The first relation is proved as follows:

$$\csc\theta = \frac{r}{y} = \frac{1}{y/r} = \frac{1}{\sin\theta}.$$

The other relations are proved similarly.

We have also:

$$(7)\ \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$(8)\ \frac{\cos\theta}{\sin\theta} = \cot\theta$$

To prove the first, we substitute and have:

$$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\theta.$$

The second is proved in similar fashion.

There remain to be discussed three other important identities:

$$(9)\ \sin^2\theta + \cos^2\theta = 1$$
$$(10)\ 1 + \tan^2\theta = \sec^2\theta$$
$$(11)\ 1 + \cot^2\theta = \csc^2\theta$$

To prove these three relations, we apply the law of Pythagoras to the triangle in Figure 35:

$$y^2 + x^2 = r^2.$$

Dividing both sides of this equation by r^2, we have:

$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1,$$

or:

$$\sin^2\theta + \cos^2\theta = 1.$$

Similarly, dividing by x^2, we obtain the second relation; and dividing by y^2, we obtain the third.

Since these relations are of fundamental importance, the student should memorize all of them.

NOTE: The object in *proving an identity* is to reduce both sides of the given relation to the same quantity. This may be done by working with the left-hand side alone, or with the right-hand side alone, or by working with both sides. In the last instance, we feel that the problem is aesthetically a bit more nicely done if the two sides are not combined;

moreover, the practice of combining the sides frequently leads to errors in computation. Thus, suppose we wish to prove the following:

$$-2 = 2.$$

Squaring, we have

$$4 = 4,$$

which is true. Hence, we reason, the original relation

$$-2 = 2$$

is true; but this conclusion is obviously absurd.

Example 1

Prove the following identity:

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \sec\theta\csc\theta.$$

Since

$$\cos^2\theta + \sin^2\theta = 1,$$

the left-hand side becomes:

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos}$$

The right-hand side becomes:

$$\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta} = \frac{1}{\sin\theta\cos\theta}.$$

Example 2

Prove:

$$(1 + \cot^2\theta)\cos^2\theta = \cot^2\theta.$$

Since

$$1 + \cot^2\theta = \csc^2\theta,$$

the left-hand side becomes:

$$\csc^2\theta\cdot\cos^2\theta = \frac{1}{\sin^2\theta}\cdot\cos^2\theta$$

$$= \frac{\cos^2\theta}{\sin^2\theta},$$

which the right-hand side also equals.

Example 3

Prove:

$$\frac{1 + \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 - \sin \theta}.$$

We know

$$1 - \sin^2 \theta = \cos^2 \theta.$$

That suggests multiplying both numerator and denominator of the left-hand side by $(1 - \sin \theta)$. We have then:

$$\begin{aligned} \frac{1 + \sin \theta}{\cos \theta} &= \frac{1 - \sin^2 \theta}{\cos \theta\,(1 - \sin \theta)} \\ &= \frac{\cos^2 \theta}{\cos \theta\,(1 - \sin \theta)} \\ &= \frac{\cos \theta}{1 - \sin \theta}, \end{aligned}$$

which the right-hand side also equals.

It is quite apparent from the above that there is no set rule to follow in proving identities; but, in general, a safe rule is:

Reduce everything to sines and cosines. Then, wherever necessary, make use of the identity: $\sin^2 \theta + \cos^2 \theta = 1$.

In later sections of the text where we are considering relations involving double-angles, half-angles, and so forth, it will generally be found desirable to reduce our quantities to functions of a *single* angle.

Of course, any time the quantity $(1 + \tan^2 \theta)$ appears, we may substitute, first,

$$\sec^2 \theta$$

and, then,

$$\frac{1}{\cos^2 \theta}.$$

If we forget that particular identity, the above method of reducing everything to sines and cosines will still hold.

Also, in general, if one side of an identity to be proved is more complicated than the other, it is advisable to reduce the more complicated side first.

Problems

Prove the following identities:

1. $\tan \theta + \cot \theta = \sec \theta \csc \theta$.
2. $\cos \theta \tan \theta = \sin \theta$.
3. $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.
4. $(\sin A - \cos A)^2 = 1 - 2 \sin A \cos A$.
5. $(\sin^2 A + \cos^2 A)^2 = 1$.
6. $(1 + \sec A)(1 - \cos A) = \tan^2 A \cos A$.
7. $\sec \theta - 1 = \sec \theta(1 - \cos \theta)$.
8. $\cos \theta + \tan \theta \sin \theta = \sec \theta$.
9. $\sin X(1 + \tan X) + \cos X(1 + \cot X) = \sec X + \csc X$.
10. $\cos X \csc X \tan X = \sin X \sec X \cot X$.
11. $\csc^4 A - \cot^4 A = \csc^2 A + \cot^2 A$.
12. $(1 + \tan \theta)(1 + \cot \theta) = (1 + \tan \theta) + (1 + \cot \theta)$.
13. $(\cos^2 \theta - 1)(\cot^2 \theta + 1) + 1 = 0$.
14. $\sin \theta \cos \theta(\sec \theta + \csc \theta) = \sin \theta + \cos \theta$.
15. $(\tan \theta - \sin \theta)^2 + (1 - \cos \theta)^2 = (\sec \theta - 1)^2$.
16. $\cot^2 X \cos^2 X = \cot^2 X - \cos^2 X$.
17. $(\sin A + \csc A)^2 + (\cos A + \sec A)^2 - \tan^2 A - \cot^2 A = 7$.
18. $\sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta$.
19. $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)$.
20. $\cos^3 \theta - \sin^3 \theta = (\cos \theta - \sin \theta)(1 + \sin \theta \cos \theta)$.
21. $1 - \tan^4 B = 2 \sec^2 B - \sec^4 B$.
22. $(\sin^2 A - \cos^2 A)^2 = 1 - 4 \cos^2 A + 4 \cos^4 A$.
23. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.
24. $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2$.
25. $\dfrac{1}{1 + \tan^2 A} + \dfrac{1}{1 + \cot^2 A} = 1$.
26. $\dfrac{\sec \theta - \csc \theta}{\sec \theta + \csc \theta} = \dfrac{\tan \theta - 1}{\tan \theta + 1}$.

27. $\dfrac{1 + 2 \cos \theta}{\sin \theta} = \csc \theta + 2 \cot \theta.$

28. $\dfrac{\tan A - 1}{\tan A + 1} = \dfrac{1 - \cot A}{1 + \cot A}.$

29. $\dfrac{1}{1 + \cos^2 A} = \dfrac{\sec^2 A}{\tan^2 A + 2}.$

30. $\dfrac{1}{\tan A + \cot A} = \sin A \cos A.$

31. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = 2 \csc A.$

32. $\tan A - \sin A = \dfrac{\sec A \sin^3 A}{1 + \cos A}.$

33. $\dfrac{1}{1 + \sin^2 A} + \dfrac{1}{1 + \cos^2 A} + \dfrac{1}{1 + \sec^2 A} + \dfrac{1}{1 + \csc^2 A} = 2.$

34. $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A.$

35. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1.$

36. $\dfrac{\tan A - \cot A}{\tan A + \cot A} = \dfrac{2}{\csc^2 A} - 1.$

37. $\dfrac{1 - \tan^2 \theta}{1 + \tan \theta} = \dfrac{\cos \theta - \sin \theta}{\cos \theta}.$

38. $\dfrac{\sin^3 X}{\cos X - \cos^3 X} = \tan X.$

39. $\dfrac{\tan X + \sec X - 1}{\tan X - \sec X + 1} = \tan X + \sec X.$

40. $\dfrac{\tan \theta - \sin \theta}{\sin^3 \theta} = \dfrac{\sec \theta}{1 + \cos \theta}.$

29. Functions of $(90° + \theta)$. For future work we wish the functions of $(90° + \theta)$ in terms of θ. We shall take θ as acute; the results obtained, however, hold when θ is of any magnitude.

In Figure 36, the angle XOB equals $(90° + \theta)$; the triangle with sides r', x', and y' appears as in the figure.

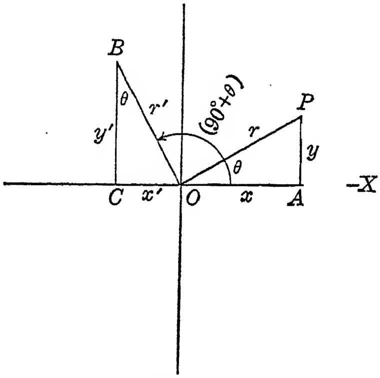

Figure 36.

From O, take OP perpendicular to OB, and of length $r = r'$. Drop a perpendicular from P to OX at A; denote by r, x, and y the sides of the right triangle thus formed. Then, the two triangles are congruent and we have: $r = r'$; $x = y'$; and $y = -x'$.

First we shall find: $\sin (90° + \theta)$.

$$\sin (90° + \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos \theta.$$

Similarly,

$$\cos (90° + \theta) = \frac{x'}{r'} = -\frac{y}{r} = -\sin \theta.$$

And:

$$\tan (90° + \theta) = -\cot \theta,$$
$$\csc (90° + \theta) = \sec \theta,$$
$$\sec (90° + \theta) = -\csc \theta,$$
$$\cot (90° + \theta) = -\tan \theta.$$

Observe that, except for the signs, the above results are exactly the same as those obtained in Section 11. Similar results obtain for the functions of $(270° \pm \theta)$. Hence we have:

Theorem. *Any function of* $(n90° \pm \theta)$ *when* n *is an odd positive integer is equal to the corresponding co-function of* θ, *with the sign depending on the quadrant in which the angle lies.*

We shall find particularly useful in Section 32 the following relation:

$$\cos (90° + \theta) = -\sin \theta.$$

30. Principal angle between two lines. Consider the two *directed* lines AB and CD, intersecting at O (Figure 37). There are various angles from the *positive* direction of one line to the *positive* direction of the other, such as those indicated by 1, 2, and 3 on the figure. Of all such angles, there is one angle which is *positive* and less than 180°. We call this angle the *principal angle*. In Figure 37, it is angle 2.

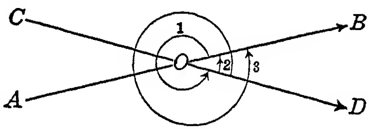

Figure 37.

31. Projection. Consider the directed line segment AB and the directed line CD in Figure 38. From A and B, respectively, drop AM and BN perpendicular to CD. The line segment MN is called the projection of AB on CD, and is written

$$\text{proj}_{CD}AB = MN.$$

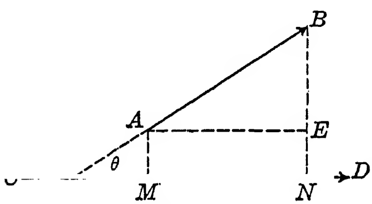

Figure 38.

Now, if AB is extended to meet CD and the principal angle is denoted by θ, and if AE is drawn perpendicular to BN, it is evident that angle $BAE = \theta$ and that $AE = MN$. Hence, since

$$\cos\theta \quad \frac{AE}{AB},$$

then:

$$MN = AE = AB\cos\theta.$$

From the above explanation we derive the first theorem on projection. The theorem is true regardless of the direction of the lines and the magnitude of θ.

$$\text{proj}_{CD}AB = AB\cos\theta$$

Theorem 1. *The projection of a line segment on any line is equal to the product of the length of the line segment and the cosine of the principal angle between the lines.*

Consider the broken line OA, AB (Figure 39). Project OA, AB, and OB on CD. Then

$$\begin{aligned}\text{proj}_{CD}OA &= MN,\\ \text{proj}_{CD}AB &= NQ \text{ (NOT: } QN),\\ \text{proj}_{CD}OB &= MQ.\end{aligned}$$

But

$$\begin{aligned}MQ &= MN - QN\\ &= MN + NQ.\end{aligned}$$

Hence:

$$\text{proj}_{CD}OB = \text{proj}_{CD}OA + \text{proj}_{CD}AB.$$

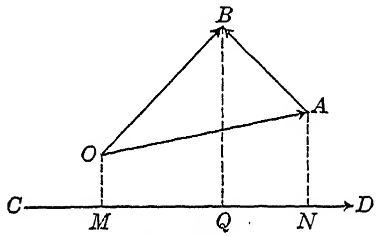

Figure 39.

From this computation, we have the second theorem on projection. This theorem may be extended for a broken line of any finite number of parts.

$$\text{proj } OB = \text{proj } OA + \text{proj } AB$$

Theorem 2. *The projection on any line of the broken line* OA, AB *is equal to the projection of* OB.

32. Sine and cosine of the sum of two angles. In the present section, we shall derive formulas for sin $(\alpha + \beta)$ and cos $(\alpha + \beta)$; α and β may be any given angles. In Figure 40, α and β are taken as acute, and are of such magnitude that their sum is less than 90°. It may be proved, however, that the formulas hold for angles of any magnitude.

Consider axes of coördinates with angles α and β at the origin O, as in Figure 40. From any point P on the terminal

side of angle β, drop a perpendicular PA to the terminal side of angle α. Extend PA to both axes to form angles as in the figure.

The right triangle OAP is the one upon which we shall focus our attention. The essence of our proof is to project the sides of this right triangle, first, on the x-axis and, then, on the y-axis. The first projection will give us $\cos(\alpha + \beta)$; the second, $\sin(\alpha + \beta)$.

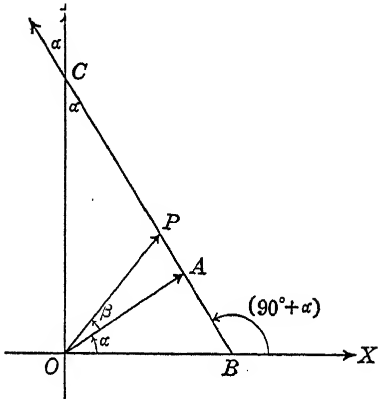

Figure 40.

Projecting the directed sides of the right triangle OAP on the x-axis, we have, by the second theorem on projection:

$$\text{proj}_{OX}OP = \text{proj}_{OX}OA + \text{proj}_{OX}AP.$$

By the first projection theorem, this becomes:

$$OP \cos(\alpha + \beta) = OA \cos\alpha + AP \cos(90° + \alpha).$$

Or, since

$$\cos(90° + \alpha) = -\sin\alpha,$$

we have:

$$OP \cos(\alpha + \beta) = OA \cos\alpha - AP \sin\alpha.$$

Dividing by OP, we have:

$$\cos(\alpha + \beta) = \cos\alpha\left(\frac{OA}{OP}\right) - \sin\alpha\left(\frac{AP}{OP}\right).$$

Or, since

$$\frac{OA}{OP} = \cos\beta$$

and

$$\frac{AP}{OP} \quad \sin\beta,$$

therefore:

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

In like fashion, we project the sides of the right triangle OAP on the y-axis and we have:

$$\text{proj}_{OY}OP = \text{proj}_{OY}OA + \text{proj}_{OY}AP.$$

Substituting,

$$OP \cos[90° - (\alpha + \beta)] = OA \cos(90° - \alpha) + AP \cos\alpha,$$

or

$$OP \sin(\alpha + \beta) = OA \sin\alpha + AP \cos\alpha.$$

Dividing by OP, we have:

$$\sin(\alpha + \beta) = \sin\alpha \left(\frac{OA}{OP}\right) + \cos\alpha \left(\frac{AP}{OP}\right).$$

Or, finally,

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

33. Tan $(\alpha + \beta)$. A formula for $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$ is derived as follows:

$$\tan(\alpha + \beta) \quad \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$
$$\frac{\sin\alpha \cos\beta + \cos\alpha \sin\beta}{\cos\alpha \cos\beta - \sin\alpha \sin\beta}.$$

Dividing each member of this last fraction by $\cos\alpha \cos\beta$, we have:

$$\tan(\alpha + \beta) = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha \sin\beta}{\cos\alpha \cos\beta}}.$$

Therefore:

$$\tan(\alpha + \beta) \quad \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

Problems

Example

Find, by using one of the addition formulas, sin 75°.

$$\begin{aligned}\sin 75^\circ &= \sin (45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

1. By using $(75^\circ = 45^\circ + 30^\circ)$, find cos 75°.

2. Find tan 75°.

3. Verify the relations for the functions of 90°.

4. Verify the relations for the functions of 180°.

5. Verify: $\cos (90^\circ + \theta) = -\sin \theta$.

6. Verify: $\sin (180^\circ + \theta) = -\sin \theta$.

7. Prove:

$$\tan (45^\circ + A) = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

8. If $\alpha + \beta = 45^\circ$, prove: $(1 + \tan \alpha)(1 + \tan \beta) = 2$.

9. Prove:

$$\cot (\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}.$$

10. If $\tan \alpha = \frac{1}{2}$ and $\tan \beta = \frac{1}{4}$, prove: $\tan (\alpha + \beta) = \frac{6}{7}$.

11. If $\tan \alpha = \frac{5}{6}$ and $\tan \beta = \frac{1}{11}$, prove: $(\alpha + \beta) = 45^\circ$ or 225°.

12. If $\tan \alpha = m$ and $\tan \beta = n$ (assuming α and β acute), prove:

$$\cos (\alpha + \beta) = \frac{1 - mn}{\sqrt{(1 + m^2)(1 + n^2)}}.$$

13. If $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{12}{5}$ (assuming α and β acute), find $\sin (\alpha + \beta)$.

14. With the material of Problem 13, find $\cos (\alpha + \beta)$.

15. If $\tan \alpha = \dfrac{m}{m+1}$ and $\tan \beta = \dfrac{1}{2m+1}$, prove:

$$\tan (\alpha + \beta) = 1.$$

34. Functions of the difference of two angles. We wish formulas for the sine, the cosine, and the tangent of $(\alpha - \beta)$. Using Roman instead of Greek letters, we rewrite

$$\sin (x + y) = \sin x \cos y + \cos x \sin y.$$

Let $x = \alpha$, and $y = -\beta$. Then, substituting, we have:

$$\sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta).$$

Or, since

$$\cos (-\beta) = \cos \beta$$

and

$$\sin (-\beta) = -\sin \beta,$$

we have:

$$\begin{aligned} \sin (\alpha - \beta) &= \sin \alpha \cos \beta + \cos \alpha (-\sin \beta) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta. \end{aligned}$$

Similar results obtain for $\cos (\alpha - \beta)$ and $\tan (\alpha - \beta)$. Hence we have:

$$\boxed{\begin{aligned} \sin (\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos (\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \tan (\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \end{aligned}}$$

Problems

1. Find:

(*a*) $\sin 15°$.
(*b*) $\cos 15°$.
(*c*) $\tan 15°$.

2. Verify:

(*a*) $\sin (180° - \theta) = \sin \theta$.
(*b*) $\cos (360° - \theta) = \cos \theta$.
(*c*) $\tan (360° - \theta) = -\tan \theta$.
(*d*) $\cos (270° - \theta) = -\sin \theta$.

3. If $\tan \alpha = \frac{1}{3}$ and $\tan \beta = \frac{1}{4}$, find $\tan (\alpha - \beta)$.

4. If $\tan \alpha = \frac{12}{13}$ and $\tan \beta = \frac{4}{5}$ (assuming α and β acute), find $\cos (\alpha - \beta)$.

5. If $\tan \alpha = (x + 1)$ and $\tan \beta = (x - 1)$, prove:

$$\cot (\alpha - \beta) = \frac{x^2}{2}.$$

6. Prove:

(a) $\tan (A - 45°) + \cot (A + 45°) = 0.$
(b) $\cot (A - 45°) + \tan (A + 45°) = 0.$
(c) $\cos (A + 45°) + \sin (A - 45°) = 0.$
(d) $\cos (A - 45°) - \sin (A + 45°) = 0.$
(e) $\sin (A - 45°) = \dfrac{\sin A - \cos A}{\sqrt{2}}.$

35. Functions of a double-angle. We wish formulas for the sine, the cosine, and the tangent of a double-angle, say 2α. Hence, we proceed as follows: Replacing β by α in our addition formula, we have:

$$\begin{aligned} \sin 2\alpha &= \sin (\alpha + \alpha) \\ &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha. \end{aligned}$$

Or:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Similarly,

$$\begin{aligned} \cos 2\alpha &= \cos (\alpha + \alpha) \\ &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha. \end{aligned}$$

Or:

$$\boxed{\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha}$$

Then, since

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

we have also:

$$\boxed{\cos 2\alpha = 2 \cos^2 \alpha - 1.}$$

Or:

$$\boxed{\cos 2\alpha = 1 - 2 \sin^2 \alpha}$$

Similarly,

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Example

Find $\sin 3A$.

$$\begin{aligned}
\sin 3A &= \sin (2A + A) \\
&= \sin 2A \cos A + \cos 2A \sin A \\
&= (2 \sin A \cos A) \cos A + (1 - 2 \sin^2 A) \sin A \\
&= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A \\
&= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A \\
&= 3 \sin A - 4 \sin^3 A
\end{aligned}$$

The above example illustrates the extensive use that may be made of the addition formulas: by this device the functions of any integral multiple of an angle may be found.

Problems

1. By using ($60° = 2 \cdot 30°$), verify values for $\sin 60°$, $\cos 60°$, and $\tan 60°$.

2. Similarly, verify the values for the above functions of $90°$.

3. Prove: $\cos 3A = 4 \cos^3 A - 3 \cos A$.

4. By using the material in Problem 3, verify the value for $\cos 90°$.

5. Given $\sin A = \frac{3}{5}$, $\cos A$ positive; find $\tan 2A$.

6. Given $\cos A = \frac{5}{13}$, $\tan A$ negative; find $\sin 2A$.

7. Given $\tan A = -\frac{2}{5}$, $\sin A$ negative; find $\cos 2A$.

8. Prove:

(*a*) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$.

(*b*) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$.

(*c*) $\dfrac{\tan (A + B) + \tan (A - B)}{1 - \tan (A + B) \tan (A - B)} = \tan 2A$.

(*d*) $\dfrac{1 + \sin 2\theta}{\cos 2\theta} = \dfrac{\cot \theta + 1}{\cot \theta - 1}$.

(e) $\tan 2A + \sec 2A = \dfrac{\cos A + \sin A}{\cos A - \sin A}$.

(f) $\csc 2\theta = \dfrac{\sec \theta \csc \theta}{2}$.

(g) $\cot A = \dfrac{\sin 2A}{1 - \cos 2A}$.

(h) $\dfrac{2 \sin^3 X}{1 - \cos X} = 2 \sin X + \sin 2X$.

(i) $\sec 2\theta = \dfrac{\sec^2 \theta}{2 - \sec^2 \theta}$.

(j) $1 + \sec 2A + \tan 2A = \dfrac{2}{1 - \tan A}$.

(k) $1 + \sin 2X = \dfrac{(1 + \tan X)^2}{1 + \tan^2 X}$.

9. Prove:

(a) $1 + \tan 2A \tan A = \sec 2A$.
(b) $\sin 2A(\tan A + \cot A) = 2$.
(c) $\tan \theta + \cot \theta = 2 \csc 2\theta$.
(d) $\cot \theta - \tan \theta = 2 \cot 2\theta$.
(e) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$.
(f) $\sin 2\alpha \sin \alpha = (1 - \cos 2\alpha) \cos \alpha$.
(g) $\cos 4X = 1 - 8 \sin^2 X + 8 \sin^4 X$.

36. Functions of a half-angle. We wish first a formula for the sine of a half angle. Let us take

$$\sin \frac{\alpha}{2}.$$

We rewrite one of our formulas for the cosine of $2X$ thus:

$$\cos 2X = 1 - 2 \sin^2 X.$$

Now let

$$X = \frac{\alpha}{2}.$$

Then

$$2X = \alpha.$$

Hence we have:

$$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2},$$

or:

$$2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha,$$

or:

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}.$$

Or, finally,

$$\boxed{\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}}$$

To find an expression for the cosine of a half-angle,

$$\cos \frac{\alpha}{2},$$

we rewrite one of the other formulas for the cosine of $2X$ thus:

$$\cos 2X = 2 \cos^2 X - 1.$$

Again let

$$X = \frac{\alpha}{2}.$$

Then

$$2X = \alpha;$$

and we have:

$$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1,$$

$$2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha,$$

$$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}.$$

Finally:

$$\boxed{\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}}$$

Next,

$$\tan \frac{\alpha}{2} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \frac{\pm \sqrt{\frac{1 - \cos \alpha}{2}}}{\pm \sqrt{\frac{1 + \cos \alpha}{2}}}.$$

Or:

$$\boxed{\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}}$$

The formula just above may be simplified as follows:

$$\begin{aligned}
\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} &= \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha} \cdot \frac{1 + \cos \alpha}{1 + \cos \alpha}} \\
&= \sqrt{\frac{1 - \cos^2 \alpha}{(1 + \cos \alpha)^2}} \\
&= \sqrt{\frac{\sin^2 \alpha}{(1 + \cos \alpha)^2}} \\
&= \frac{\sin \alpha}{1 + \cos \alpha}.
\end{aligned}$$

Similarly, if we multiply both numerator and denominator of

$$\frac{1 - \cos \alpha}{1 + \cos \alpha}$$

by

$$1 - \cos \alpha,$$

we obtain:

$$\frac{1 - \cos \alpha}{\sin \alpha}.$$

Hence we have:

$$\boxed{\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}}$$

Example

Prove:

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}.$$

We might solve this problem as follows:

$$\begin{aligned}
2 \sin \frac{A}{2} \cos \frac{A}{2} &= 2 \cdot \pm\sqrt{\frac{1 - \cos A}{2}} \cdot \pm\sqrt{\frac{1 + \cos A}{2}} \\
&= 2 \cdot \pm\sqrt{\frac{1 - \cos^2 A}{4}} \\
&= 2 \cdot \pm\sqrt{\frac{\sin^2 A}{4}} \\
&= 2 \cdot \frac{\sin A}{2} \\
&= \sin A.
\end{aligned}$$

However, at this point the skillful student will recognize that the original relation is the formula (disguised a bit) for the sine of a double-angle. If we let

$$A = 2X,$$

then

$$\frac{A}{2} = X;$$

and the relation

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

becomes:

$$\sin 2X = 2 \sin X \cos X.$$

Indeed,

$$\sin \frac{A}{2} = 2 \sin \frac{A}{4} \cos \frac{A}{4}$$

is true for the same reason.

In other words, if an equation assumes the *form* of a well-known formula and the angles have the proper relation,

one to another, the equation is true, regardless of the form in which the angles are expressed. Thus

$$\cos\theta \quad 1 - 2\sin^2\frac{\theta}{2}$$

will be recognized as one of our formulas for $\cos 2\alpha$.

The student should not be misled, however, into thinking that all the problems below are solved in a similar manner. They are not. The above fact was pointed out simply as being of use in some instances only.

Problems

1. Given $\sin A = \frac{3}{5}$, $\tan A$ positive; find $\sin\frac{A}{2}$.

2. Given $\cos A = \frac{2}{3}$, $\sin A$ negative; find $\tan\frac{A}{2}$.

3. Given $\tan A = -\frac{5}{12}$, A not in the second quadrant; find $\cos\frac{A}{2}$.

4. Given $\csc A = -\frac{5}{4}$, A not in the fourth quadrant; find $\tan\frac{A}{2}$.

5. Given $\cot A = \frac{4}{3}$, A not in the second quadrant; find $\sin\frac{A}{2}$.

6. Prove the following identities:

(*a*) $1 + \tan A \tan\frac{A}{2} = \sec A$.

(*b*) $\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{1 + \sin A}$.

(*c*) $\dfrac{1}{\csc A - \cot A} = \cot\frac{A}{2}$.

(*d*) $\dfrac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \cos\theta$.

(*e*) $\tan^2\frac{A}{2} = \dfrac{2\sin A - \sin 2A}{2\sin A + \sin 2A}$.

(*f*) $\cot\frac{A}{2} = \dfrac{\sin A + \sin 2A}{\cos A - \cos 2A}$.

(g) $\dfrac{1+\tan (A/2)}{1-\tan (A/2)} = \tan A + \sec A.$

(h) $\left(\cot \dfrac{X}{2} - \tan \dfrac{X}{2}\right)^2\left(\cot X - 2 \cot 2X\right) = 4 \cot X.$

(i) $\left(1 + \cot^2 \dfrac{A}{2}\right) \sin A \tan \dfrac{A}{2} = 2.$

(j) $\sin^2 \dfrac{\theta}{2}\left(\cot \dfrac{\theta}{2} - 1\right)^2 = 1 - \sin \theta.$

(k) $\tan \dfrac{A}{2} \tan A + 1 = \tan A \cot \dfrac{A}{2} - 1.$

(l) $\tan \left(45° + \dfrac{\theta}{2}\right) = \cot \left(45° - \dfrac{\theta}{2}\right).$

(m) $\cot \dfrac{A}{2} - \tan \dfrac{A}{2} = 2 \cot \dfrac{A}{2}.$

(n) $\tan \dfrac{A}{2} = \dfrac{1 - \cos A + \sin A}{1 + \cos A + \sin A}.$

(o) $1 - 2 \cot X \tan \dfrac{X}{2} - \tan^2 \dfrac{X}{2} = 0.$

37. Product formulas. We wish a formula for

$$\sin P + \sin Q$$

expressed as a product. We proceed as follows: Let $P = \alpha + \beta$, and $Q = \alpha - \beta$. Then:

$$\begin{aligned} \sin P + \sin Q &= \sin (\alpha + \beta) + \sin (\alpha - \beta) \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= 2 \sin \alpha \cos \beta. \end{aligned}$$

Here is our product; but it is in terms of functions of α and β, and we wish it to involve P and Q. Solving the original relations for α and β by addition and subtraction, we have:

$$\alpha = \frac{P + Q}{2},$$

$$\beta = \frac{P - Q}{2}.$$

Hence we have:

$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

In like manner, we derive the other three product formulas and collect the four below:

$$\boxed{\begin{aligned}
&(1)\ \sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}\\
&(2)\ \sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}\\
&(3)\ \cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}\\
&(4)\ \cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}
\end{aligned}}$$

The above formulas are particularly useful in the branch of mathematics called calculus.

Example 1

Prove:

$$\sin 3A - \sin A = 2 \cos 2A \sin A.$$

Let

$$3A = P,$$

$$A = Q.$$

Then

$$\frac{P+Q}{2} = \frac{4A}{2} = 2A,$$

and

$$\frac{P-Q}{2} = \frac{3A-A}{2} = A.$$

Then, substituting in the second of the product formulas, we have immediately:

$$\sin 3A - \sin A = 2 \cos 2A \sin A.$$

Example 2

Prove:

$$\frac{\sin 3A + \sin 5A}{\cos 3A - \cos 5A} = \cot A.$$

Let
$$P = 3A,$$
$$Q = 5A.$$

Then
$$\frac{P+Q}{2} = 4A,$$

and
$$\frac{P-Q}{2} = -A.$$

When we substitute the first product formula, the numerator becomes:

$$\begin{aligned}\sin 3A + \sin 5A &= 2 \sin 4A \cos (-A)\\ &= 2 \sin 4A \cos A.\end{aligned}$$

When we substitute the fourth product formula, the denominator becomes:

$$\begin{aligned}\cos 3A - \cos 5A &= -2 \sin 4A \sin (-A)\\ &= -2 \sin 4A(- \sin A)\\ &= 2 \sin 4A \sin A.\end{aligned}$$

Hence:
$$\begin{aligned}\frac{\sin 3A + \sin 5A}{\cos 3A - \cos 5A} &= \frac{2 \sin 4A \cos A}{2 \sin 4A \sin A}\\ &= \frac{\cos A}{\sin A}\\ &= \cot A.\end{aligned}$$

Problems

Prove the following identities:

1. $\sin 5A + \sin 3A = 2 \sin 4A \cos A.$
2. $\cos 4A - \cos 2A = -2 \sin 3A \sin A.$
3. $\sin 2A + \sin 4A + \sin 6A = 4 \cos A \cos 2A \sin 3A.$
4. $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A.$
5. $\dfrac{\sin 2A + \sin A}{\cos 2A + \cos A} = \tan \dfrac{3A}{2}.$
6. $\dfrac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} = \tan \dfrac{A}{2}.$
7. $\dfrac{\sin A + \sin B}{\cos A - \cos B} = \dfrac{\cos A + \cos B}{\sin B - \sin A}.$

8. $\dfrac{\cos 6A - \cos 4A}{\sin 6A + \sin 4A} = -\tan A.$

9. $\dfrac{\sin 7A + \sin 3A}{\cos 7A - \cos 3A} = -\cot 2A.$

10. $\dfrac{\sin 5A - \sin A}{\cos 5A + \cos A} = \tan 2A.$

11. $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \dfrac{1}{\sqrt{3}}.$

12. $\dfrac{\sin 5A - 2\sin 3A + \sin A}{\cos 5A - 2\cos 3A + \cos A} = \tan 3A.$

13. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$

14. $\dfrac{\sin 8A - \sin 6A + \sin 4A - \sin 2A}{\cos 8A - \cos 6A + \cos 4A - \cos 2A} = -\cot 5A.$

15. $\dfrac{\sin 3A + \sin 2A + \sin A}{\cos 3A + \cos 2A + \cos A} = \tan 2A.$

16. $\dfrac{\sin (A + 2B) - 2\sin (A + B) + \sin A}{\cos (A + 2B) - 2\cos (A + B) + \cos A} = \tan (A + B).$

17. $\dfrac{\sin (2A - 3B) + \sin 3B}{\cos (2A - 3B) + \cos 3B} = \tan A.$

18. $\dfrac{\sin 47^\circ + \sin 73^\circ}{\cos 47^\circ + \cos 73^\circ} = \sqrt{3}.$

19. $\dfrac{\sin 4A - \sin 2A}{\sin 4A + \sin 2A} = \dfrac{1 - 3\tan^2 A}{3 - \tan^2 A}.$

20. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \dfrac{A + B}{2}}{\tan \dfrac{A - B}{2}}.$

Miscellaneous Problems

Prove the following identities:

1. $\sin 5A \sin A = \sin^2 3A - \sin^2 2A.$
2. $\sin \theta + \sin (\theta - 120^\circ) + \sin (60^\circ - \theta) = 0.$
3. $\cos^4 \theta - \sin^4 \theta = 2\cos^2 \theta - 1.$
4. $(\sec A - \tan A)(\sec A + \tan A) = 1.$

5. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta.$

6. $\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta.$

7. $\sin 4X = 8\cos^3 X\sin X - 4\cos X\sin X.$

8. $\dfrac{\sin A + \cos A}{\cos A - \sin A} = \tan 2A + \sec 2A.$

9. $\cot^2\theta\left(\dfrac{\sec\theta - 1}{1 + \sin\theta}\right) + \sec^2\theta\left(\dfrac{\sin\theta - 1}{1 + \sec\theta}\right) = 0.$

10. $\dfrac{\tan^3 A}{1 + \tan^2 A} + \dfrac{\cot^3 A}{1 + \cot^2 A} = \dfrac{1 - 2\sin^2 A\cos^2 A}{\sin A\cos A}.$

11. $\dfrac{1 + \sin 2X + \cos 2X}{1 + \sin 2X - \cos 2X} = \cot X.$

12. $\dfrac{\sin(A - B)}{\sin A\sin B} + \dfrac{\sin(B - C)}{\sin B\sin C} + \dfrac{\sin(C - A)}{\sin C\sin A} = 0.$

13. $\cos^6\theta - \sin^6\theta = \cos 2\theta\left(1 - \dfrac{\sin^2 2\theta}{4}\right).$

14. $\cos^4 X + \sin^4 X = 1 - \dfrac{\sin^2 2X}{2}.$

15. $\dfrac{2}{(1 + \tan\theta)(1 + \cot\theta)} = \dfrac{\sin 2\theta}{1 + \sin 2\theta}.$

16. $\dfrac{\sin(A - C) + 2\sin A + \sin(A + C)}{\sin(B - C) + 2\sin B + \sin(B + C)} = \dfrac{\sin A}{\sin B}.$

17. $\dfrac{1 + \cos A}{1 - \cos A} = (\csc A + \cot A)^2.$

18. $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2.$

19. $\dfrac{\tan A}{\tan A - \tan 3A} + \dfrac{\cot A}{\cot A - \cot 3A} = 1.$

20. $\dfrac{2\sin 2A - \sin 4A}{2\sin 2A + \sin 4A} = \tan^2 A.$

21. $\dfrac{1 - \cos X}{\sin X} = \dfrac{\sin 2X}{2\cos X + \cos 2X + 1}.$

22. $\dfrac{2\sec^2 X}{2\tan X + 1} - \dfrac{\sec^2 X}{\tan X + 2} = \dfrac{6}{4 + 5\sin 2X}.$

23. $\cos\dfrac{3A}{2} = \cos\dfrac{A}{2}(2\cos A - 1).$

24. $$\frac{4 \sin \theta \cos \frac{\theta}{2}}{2 \sin \theta + \sin 2\theta} = \sec \frac{\theta}{2}.$$

25. $$\frac{\sin A + \cos B}{\sin A - \cos B} = \frac{\tan \left(\frac{A + B}{2} + 45^\circ\right)}{\tan \left(\frac{A - B}{2} - 45^\circ\right)}.$$

CHAPTER VII

SUPPLEMENTARY TOPICS*

38. Law of tangents. In Section 27 when we were considering the solution of an oblique triangle by means of the law of cosines, it was pointed out that there were additional formulas better adapted to logarithmic use but that we felt them to be unnecessary. However, since some authorities prefer them, we shall derive such formulas and include them in this chapter.

The first of these formulas is called the *law of tangents.* We shall proceed to its derivation.

Given an oblique triangle ABC, we have, from the law of sines,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

By the theory of proportion, this becomes:

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

From the first two product formulas, the right-hand side becomes:

$$\frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$

Or, we have the *law of tangents:*

* This chapter may be omitted if the instructor does not wish to include the material in his course.

$$\boxed{\frac{a-b}{a+b} = \frac{\tan \dfrac{A-B}{2}}{\tan \dfrac{A+B}{2}}}$$

By use of the law of tangents, we can solve a triangle if two sides and the included angle are given, as in the example below.

Example

Solve the triangle ABC; given $a = 2439$, $b = 1036$, $C = 38° 7'$.

$$a - b = 1403$$

$$a + b = 3475$$

$$A + B = 180° - C = 180° - 38° 8' = 141° 52'$$

$$\frac{A+B}{2} = 70° 56'$$

$$\therefore \tan \frac{A-B}{2} = \frac{(a-b)\tan \dfrac{A+B}{2}}{a+b}$$

Using logarithms, we continue:

$$\log \tan \frac{A-B}{2} = \log (a-b) + \log \tan \frac{A+B}{2} - \log (a+b)$$

$$= \log 1403 + \log \tan 70° 56' - \log 3475$$

$$\begin{aligned}
\log 1403 &= 3.1470 \\
\log \tan 70° 56' &= \underline{.4614} \\
\log \text{numerator} &= 3.6084 \\
\log 3475 &= \underline{3.5410} \\
\log \tan \frac{A-B}{2} &= .0674 \\
\frac{A-B}{2} &= 49° 26'
\end{aligned}$$

But, since $\qquad \dfrac{A+B}{2} = \underline{70° 56'}$

by addition, $\qquad \therefore A = 120° 22'$;

and, by subtraction, $\therefore B = 21° 30'$.

Side c may now be found by the law of sines.

Problems

Using the law of tangents, solve the following triangles:

1. $a = 28.43$, $b = 16.92$, $C = 40° 9'$.
2. $b = 623.1$, $c = 420.3$, $A = 62° 42'$.
3. $c = 53.28$, $a = 33.93$, $B = 63° 24'$.
4. $a = 419.2$, $b = 300.3$, $C = 53° 18'$.
5. $a = 60.66$, $b = 70.34$, $C = 46° 26'$.

39. Tangent of a half-angle in terms of the sides of a given triangle. We shall now derive formulas to be used in solving a triangle when three sides are given.

Given the triangle ABC. Let

$$\boxed{\frac{a+b+c}{2} = s}$$

then:

$$s - a = \frac{b+c-a}{2},$$

$$s - b = \frac{c+a-b}{2},$$

$$s - c = \frac{a+b-c}{2}.$$

By the tan $\frac{A}{2}$ formula and the law of cosines:

$$\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$= \sqrt{\frac{1 - \dfrac{b^2+c^2-a^2}{2bc}}{1 + \dfrac{b^2+c^2-a^2}{2bc}}}.$$

Or, by substitution,

$$\tan\frac{A}{2} = \sqrt{\frac{2bc - b^2 - c^2 + a^2}{2bc + b^2 + c^2 - a^2}}$$

$$= \sqrt{\frac{a^2 - (b - c)^2}{(b + c)^2 - a^2}}$$

$$= \sqrt{\frac{(a - b + c)(a + b - c)}{(b + c + a)(b + c - a)}}$$

$$= \sqrt{\frac{(2)(s - b)(2)(s - c)}{(2s)(2)(s - a)}}.$$

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

To make this expression more symmetrical, we write it:

$$\tan\frac{A}{2} = \sqrt{\frac{(s - a)(s - b)(s - c)}{s(s - a)^2}},$$

or

$$\tan\frac{A}{2} = \frac{1}{s - a}\sqrt{\frac{(s - a)(s - b)(s - c)}{s}}.$$

Now let

$$\boxed{\sqrt{\frac{(s - a)(s - b)(s - c)}{s}} = r}$$

and we have:

$$\boxed{\tan\frac{A}{2} = \frac{r}{s - a}}$$

Similarly,

$$\boxed{\tan\frac{B}{2} = \frac{r}{s - b}}$$

$$\boxed{\tan\frac{C}{2} = \frac{r}{s - c}}$$

We shall next derive a formula for the area of the triangle ABC in terms of the sides. From Problem 2 in Section 27, we found:

$$\text{area} = \frac{1}{2} bc \sin A.$$

Or:

$$\begin{aligned}
(\text{area})^2 &= \frac{1}{4} (bc)^2 \sin^2 A \\
&= \frac{1}{4} (bc)^2(1 - \cos^2 A) \\
&= \frac{1}{4} (bc)^2(1 + \cos A)(1 - \cos A).
\end{aligned}$$

Or:

$$\begin{aligned}
\text{area} &= \frac{1}{2} bc \sqrt{(1 + \cos A)(1 - \cos A)} \\
&= \frac{1}{2} bc \sqrt{\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)} \\
&= \frac{1}{2} bc \sqrt{\frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4(bc)^2}} \\
&= \sqrt{s(s - a)(s - b)(s - c)}.
\end{aligned}$$

Therefore:

$$\text{area} = \sqrt{s(s - a)(s - b)(s - c)}$$

Example

Solve the triangle ABC; given $a = 100$, $b = 120$, $c = 140$.

$$\begin{aligned}
2s &= 360 \\
s &= 180 \\
s - a &= 80 \\
s - b &= 60 \\
s - c &= 40
\end{aligned}$$

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

$$\log r = \frac{1}{2}\left[\log (s-a) + \log (s-b) + \log (s-c) - \log s\right]$$

$$\begin{aligned}
\log (s-a) &= \log 80 = 1.9031 \\
\log (s-b) &= \log 60 = 1.7782 \\
\log (s-c) &= \log 40 = \underline{1.6021} \\
\log \text{sum} &= 5.2834 \\
\log s = \log 180 &= \underline{2.2553} \\
2 \log r &= 3.0281 \\
\therefore \log r &= 1.5141
\end{aligned}$$

$$\begin{aligned}
\log \tan \frac{A}{2} &= \log r - \log (s-a) \\
&= (11.5141 - 10) - 1.9031 \\
&= 9.6110 - 10 \\
\frac{A}{2} &= 22^\circ\, 13' \\
\therefore A &= 44^\circ\, 26'
\end{aligned}$$

$$\begin{aligned}
\log \tan \frac{B}{2} &= \log r - \log (s-b) \\
&= (11.5141 - 10) - 1.7782 \\
&= 9.7359 - 10 \\
\frac{B}{2} &= 28^\circ\, 34' \\
\therefore B &= 57^\circ\, 8'
\end{aligned}$$

$$\begin{aligned}
\log \tan \frac{C}{2} &= \log r - \log (s-c) \\
&= (11.5141 - 10) - 1.6021 \\
&= 9.9120 - 10 \\
\frac{C}{2} &= 39^\circ\, 14' \\
\therefore C &= 78^\circ\, 28'
\end{aligned}$$

Proof

$$\begin{aligned}
A + B + C &= 44^\circ\, 26' + 57^\circ\, 8' + 78^\circ\, 28' \\
&= 179^\circ\, 62' \\
&= 180^\circ\, 2'.
\end{aligned}$$

$$\log \text{area} = \frac{1}{2}\left[\log (s - a) + \log (s - b) + \log (s - c) + \log s\right]$$
$$= \frac{1}{2}(7.5387)$$
$$\log \text{area} = 3.76985$$
$$\therefore \text{area} = 5887 \text{ square units}$$

Problems

Solve the following triangles, and find the area of each:

1. $a = 20.34$, $b = 16.48$, $c = 30.24$.
2. $a = 144$, $b = 266$, $c = 300$.
3. $a = 2743$, $b = 3201$, $c = 4002$.
4. $a = 200$, $b = 400$, $c = 500$.
5. $a = 42.81$, $b = 22.03$, $c = 30.22$.

40. Radius of the inscribed circle. It is interesting to note that the quantity

$$\sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

associated with the triangle ABC is actually the numerical length of the radius of the circle inscribed in the given triangle. We prove this result as follows:

Proof

From plane geometry, we know:

$$\text{area of } \Delta ABC = \frac{1}{2}r' \cdot P,$$

where r' is the radius of the inscribed circle and P is the perimeter of the triangle.

Since $\qquad P = 2s,$

we have: $\qquad \text{area} = r' \cdot s.$

From the formula for area derived in Section 39, we have:

$$\text{area} = \sqrt{(s - a)(s - b)(s - c)s}$$
$$= \sqrt{\frac{(s - a)(s - b)(s - c)s^2}{s}}$$

$$s\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= s \cdot r.$$

$$\therefore r' = r.$$

We give below another proof that is independent of area. Consider the triangle ABC—with inscribed circle, and with bisectors of the angles meeting at the center—in Figure 41.

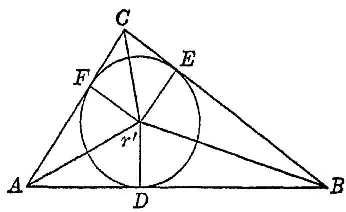

Figure 41.

Let r' be the radius of the circle. Then

$$\tan\frac{A}{2} = \frac{r'}{AD}.$$

We wish to prove: $AD = s - a$.

Proof

From plane geometry, we have: $AD = AF$, $DB = BE$, $CF = CE$.

Then $\qquad AD + DB + CF = s.$

Hence: $\qquad AD = s - (DB + CF)$

$$= s - (BE + CE);$$

or by substitution, $\qquad = s - a.$

Therefore: $\qquad \tan\frac{A}{2} = \frac{r'}{s-a}.$

But since we know $\qquad \tan\frac{A}{2} = \frac{r}{s-a},$

$$\therefore r' = r.$$

41. Circular measure of an angle. Frequently it is desirable, particularly in calculus, to express angles in units other than degrees. We do so by means of circular measure, as illustrated by Figure 42.

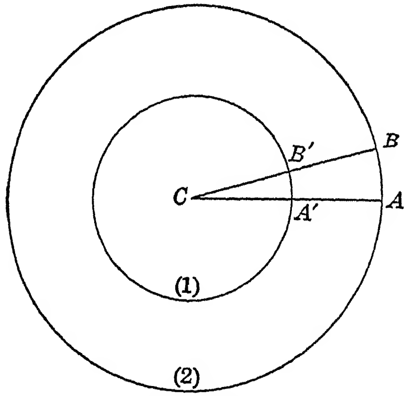

Figure 42.

Consider angle ACB and circles (1) and (2), with center at C and with radii r_1 and r_2. Since

$$\frac{A'B'}{AB} = \frac{r_1}{r_2},$$

we have:

$$\frac{A'B'}{r_1} \quad \frac{AB}{r_2}.$$

In other words, the ratio

$$\frac{\text{length of arc}}{\text{radius of circle}}$$

is always the *same* for a given central angle. We call this ratio the *circular measure of an angle*, and we call the unit a *radian*. Hence we have:

$$\begin{matrix}\text{number of radians}\\ \text{in central angle}\end{matrix} = \frac{\text{length of arc}}{\text{length of radius}}$$

To find the size of one radian, we substitute in the above formula and then have:

$$1 \text{ radian} = \frac{\text{length of arc}}{\text{length of radius}}$$

Hence:

$$\text{length of arc} = \text{length of radius.}$$

It is therefore evident that a radian is a central angle subtended by an arc equal in length to the radius of a circle.

There are as many radians in 360° as there are arcs of length r in a complete circumference. Since the circumference equals $2\pi r$, there are 2π such arcs. Consequently there are 2π radians in 360°; or:

$$\pi \text{ radians} = 180°$$

Hence, to change degrees to radians, multiply the degrees by

$$\frac{\pi}{180}.$$

To change radians to degrees, multiply the radians by

$$\frac{180}{\pi}.$$

Thus:

$$60° = 60 \cdot \frac{\pi}{180} \text{ radians} = \frac{\pi}{3} \text{ radians.}$$

Also:

$$\frac{2\pi}{3} \text{ radians} = \frac{2\pi}{3} \cdot \frac{180}{\pi} \text{ degrees} = 120°$$

Since a radian is equivalent to

$$\frac{180°}{3.14159},$$

a radian equals approximately

$$57°\ 17'\ 45''.$$

Similarly, one degree equals

$$.01745\ .\ .\ .\ \text{radian}.$$

Problems

1. Change from degrees to radians:

(*a*) 60°. (*b*) 30°. (*c*) 120°. (*d*) 45°.
(*e*) 270°. (*f*) 240°. (*g*) 360°. (*h*) 0°.
(*i*) 150°. (*j*) 300°. (*k*) 180°. (*l*) 90°.

2. Change from radians to degrees:

(*a*) $\frac{\pi}{2}$. (*b*) $\frac{2\pi}{3}$. (*c*) $\frac{3\pi}{4}$. (*d*) 2π.
(*e*) $\frac{\pi}{6}$. (*f*) $\frac{\pi}{3}$. (*g*) $\frac{5\pi}{3}$. (*h*) 4π.
(*i*) $\frac{3\pi}{5}$. (*j*) $\frac{\pi}{4}$. (*k*) $\frac{4\pi}{5}$. (*l*) 3π.

42. Summary of trigonometric formulas. For convenience, we have collected in summary outline form the trigonometric formulas developed in the preceding text.

1. Fundamental Identities.

$$\csc A \quad \frac{1}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\cot A = \frac{1}{\tan A}$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A = \frac{\cos A}{\sin A}$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \csc^2 A$$

2. Addition and Subtraction Formulas.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

3. Double-Angle Formulas.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$\cos 2\alpha = 1 - 2\sin^2\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

4. Half-Angle Formulas.

$$\sin\frac{\alpha}{2} \qquad 1 - \cos\alpha$$
$$\cos\frac{\alpha}{2} = \pm \qquad 1 + \cos\alpha$$
$$\tan\frac{\alpha}{2} = \pm \quad \frac{1 - \cos\alpha}{1 + \cos\alpha}$$
$$\tan\frac{\alpha}{2} \quad \frac{\sin\alpha}{1 + \cos\alpha}$$
$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}$$

5. Product Formulas.

$$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$

$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

6. Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

7. Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

8. Law of Tangents.

$$\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$$

9. Semi-Perimeter Formulas.

$$\tan \frac{A}{2} = \frac{r}{s-a}$$

$$\tan \frac{B}{2} = \frac{r}{s-b}$$

$$\tan \frac{C}{2} = \frac{r}{s-c}$$

$$s = \frac{a+b+c}{2}$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

10. Circular Measure.

$$\theta = \frac{l}{r}$$

The terms in this formula are interpreted: θ = number of radians in a central angle, l = length of intercepted arc, r = length of radius.

$$\pi \text{ radians} = 180°$$

11. Laws of Logarithms.

$$\log_b AB = \log_b A + \log_b B$$

$$\log_b \frac{A}{B} = \log_b A - \log_b B$$

$$\log_b A^n = n \log_b A$$

12. Projection Theorems.

$$\text{proj } AB + \text{proj } BC = \text{proj } AC$$

$$\text{proj}_{CD}\, AB = AB \cos \theta$$

The second theorem holds where θ is the principal angle between AB and CD.

Table I

LOGARITHMS TO FOUR PLACES

FOUR-PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

TABLE II

TRIGONOMETRIC FUNCTIONS TO FOUR PLACES

FOUR-PLACE TRIGONOMETRIC FUNCTIONS

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
0° 00′	.0000	———	.0000	———	———	———	1.0000	.0000	**90° 00′**
10	.0029	7.4637	.0029	7.4637	343.77	2.5363	1.0000	.0000	50
20	.0058	7.7648	.0058	7.7648	171.89	2.2352	1.0000	.0000	40
30	.0087	7.9408	.0087	7.9409	114.59	2.0591	1.0000	.0000	30
40	.0116	8.0658	.0116	8.0658	85.940	1.9342	.9999	.0000	20
50	.0145	8.1627	.0145	8.1627	68.750	1.8373	.9999	.0000	10
1° 00′	.0175	8.2419	.0175	8.2419	57.290	1.7581	.9998	9.9999	**89° 00′**
10	.0204	8.3088	.0204	8.3089	49.104	1.6911	.9998	9.9999	50
20	.0233	8.3668	.0233	8.3669	42.964	1.6331	.9997	9.9999	40
30	.0262	8.4179	.0262	8.4181	38.188	1.5819	.9997	9.9999	30
40	.0291	8.4637	.0291	8.4638	34.368	1.5362	.9996	9.9998	20
50	.0320	8.5050	.0320	8.5053	31.242	1.4947	.9995	9.9998	10
2° 00′	.0349	8.5428	.0349	8.5431	28.636	1.4569	.9994	9.9997	**88° 00′**
10	.0378	8.5776	.0378	8.5779	26.432	1.4221	.9993	9.9997	50
20	.0407	8.6097	.0407	8.6101	24.542	1.3899	.9992	9.9996	40
30	.0436	8.6397	.0437	8.6401	22.904	1.3599	.9990	9.9996	30
40	.0465	8.6677	.0466	8.6682	21.470	1.3318	.9989	9.9995	20
50	.0494	8.6940	.0495	8.6945	20.206	1.3055	.9988	9.9995	10
3° 00′	.0523	8.7188	.0524	8.7194	19.081	1.2806	.9986	9.9994	**87° 00′**
10	.0552	8.7423	.0553	8.7429	18.075	1.2571	.9985	9.9993	50
20	.0581	8.7645	.0582	8.7652	17.169	1.2348	.9983	9.9993	40
30	.0610	8.7857	.0612	8.7865	16.350	1.2135	.9981	9.9992	30
40	.0640	8.8059	.0641	8.8067	15.605	1.1933	.9980	9.9991	20
50	.0669	8.8251	.0670	8.8261	14.924	1.1739	.9978	9.9990	10
4° 00′	.0698	8.8436	.0699	8.8446	14.301	1.1554	.9976	9.9989	**86° 00′**
10	.0727	8.8613	.0729	8.8624	13.727	1.1376	.9974	9.9989	50
20	.0756	8.8783	.0758	8.8795	13.197	1.1205	.9971	9.9988	40
30	.0785	8.8946	.0787	8.8960	12.706	1.1040	.9969	9.9987	30
40	.0814	8.9104	.0816	8.9118	12.251	1.0882	.9967	9.9986	20
50	.0843	8.9256	.0846	8.9272	11.826	1.0728	.9964	9.9985	10
5° 00′	.0872	8.9403	.0875	8.9420	11.430	1.0580	.9962	9.9983	**85° 00′**
10	.0901	8.9545	.0904	8.9563	11.059	1.0437	.9959	9.9982	50
20	.0929	8.9682	.0934	8.9701	10.712	1.0299	.9957	9.9981	40
30	.0958	8.9816	.0963	8.9836	10.385	1.0164	.9954	9.9980	30
40	.0987	8.9945	.0992	8.9966	10.078	1.0034	.9951	9.9979	20
50	.1016	9.0070	.1022	9.0093	9.7882	.9907	.9948	9.9977	10
6° 00′	.1045	9.0192	.1051	9.0216	9.5144	.9784	.9945	9.9976	**84° 00′**
10	.1074	9.0311	.1080	9.0336	9.2553	.9664	.9942	9.9975	50
20	.1103	9.0426	.1110	9.0453	9.0098	.9547	.9939	9.9973	40
30	.1132	9.0539	.1139	9.0567	8.7769	.9433	.9936	9.9972	30
40	.1161	9.0648	.1169	9.0678	8.5555	.9322	.9932	9.9971	20
50	.1190	9.0755	.1198	9.0786	8.3450	.9214	.9929	9.9969	10
7° 00′	.1219	9.0859	.1228	9.0891	8.1443	.9109	.9925	9.9968	**83° 00′**
10	.1248	9.0961	.1257	9.0995	7.9530	.9005	.9922	9.9966	50
20	.1276	9.1060	.1287	9.1096	7.7704	.8904	.9918	9.9964	40
30	.1305	9.1157	.1317	9.1194	7.5958	.8806	.9914	9.9963	30
40	.1334	9.1252	.1346	9.1291	7.4287	.8709	.9911	9.9961	20
50	.1363	9.1345	.1376	9.1385	7.2687	.8615	.9907	9.9959	10
8° 00′	.1392	9.1436	.1405	9.1478	7.1154	.8522	.9903	9.9958	**82° 00′**
10	.1421	9.1525	.1435	9.1569	6.9682	.8431	.9899	9.9956	50
20	.1449	9.1612	.1465	9.1658	6.8269	.8342	.9894	9.9954	40
30	.1478	9.1697	.1495	9.1745	6.6912	.8255	.9890	9.9952	30
40	.1507	9.1781	.1524	9.1831	6.5606	.8169	.9886	9.9950	20
50	.1536	9.1863	.1554	9.1915	6.4348	.8085	.9881	9.9948	10
9° 00′	.1564	9.1943	.1584	9.1997	6.3138	.8003	.9877	9.9946	**81° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Degrees
	Cosine		Cotangent		Tangent		Sine		

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
9° 00′	.1564	9.1943	.1584	9.1997	6.3138	.8003	.9877	9.9946	**81° 00′**
10	.1593	9.2022	.1614	9.2078	6.1970	.7922	.9872	9.9944	50
20	.1622	9.2100	.1644	9.2158	6.0844	.7842	.9868	9.9942	40
30	.1650	9.2176	.1673	9.2236	5.9758	.7764	.9863	9.9940	30
40	.1679	9.2251	.1703	9.2313	5.8708	.7687	.9858	9.9938	20
50	.1708	9.2324	.1733	9.2389	5.7694	.7611	.9853	9.9936	10
10° 00′	.1736	9.2397	.1763	9.2463	5.6713	.7537	.9848	9.9934	**80° 00′**
10	.1765	9.2468	.1793	9.2536	5.5764	.7464	.9843	9.9931	50
20	.1794	9.2538	.1823	9.2609	5.4845	.7391	.9838	9.9929	40
30	.1822	9.2606	.1853	9.2680	5.3955	.7320	.9833	9.9927	30
40	.1851	9.2674	.1883	9.2750	5.3093	.7250	.9827	9.9924	20
50	.1880	9.2740	.1914	9.2819	5.2257	.7181	.9822	9.9922	10
11° 00′	.1908	9.2806	.1944	9.2887	5.1446	.7113	.9816	9.9919	**79° 00′**
10	.1937	9.2870	.1974	9.2953	5.0658	.7047	.9811	9.9917	50
20	.1965	9.2934	.2004	9.3020	4.9894	.6980	.9805	9.9914	40
30	.1994	9.2997	.2035	9.3085	4.9152	.6915	.9799	9.9912	30
40	.2022	9.3058	.2065	9.3149	4.8430	.6851	.9793	9.9909	20
50	.2051	9.3119	.2095	9.3212	4.7729	.6788	.9787	9.9907	10
12° 00′	.2079	9.3179	.2126	9.3275	4.7046	.6725	.9781	9.9904	**78° 00′**
10	.2108	9.3238	.2156	9.3336	4.6382	.6664	.9775	9.9901	50
20	.2136	9.3296	.2186	9.3397	4.5736	.6603	.9769	9.9899	40
30	.2164	9.3353	.2217	9.3458	4.5107	.6542	.9763	9.9896	30
40	.2193	9.3410	.2247	9.3517	4.4494	.6483	.9757	9.9893	20
50	.2221	9.3466	.2278	9.3576	4.3897	.6424	.9750	9.9890	10
13° 00′	.2250	9.3521	.2309	9.3634	4.3315	.6366	.9744	9.9887	**77° 00′**
10	.2278	9.3575	.2339	9.3691	4.2747	.6309	.9737	9.9884	50
20	.2306	9.3629	.2370	9.3748	4.2193	.6252	.9730	9.9881	40
30	.2334	9.3682	.2401	9.3804	4.1653	.6196	.9724	9.9878	30
40	.2363	9.3734	.2432	9.3859	4.1126	.6141	.9717	9.9875	20
50	.2391	9.3786	.2462	9.3914	4.0611	.6086	.9710	9.9872	10
14° 00′	.2419	9.3837	.2493	9.3968	4.0108	.6032	.9703	9.9869	**76° 00′**
10	.2447	9.3887	.2524	9.4021	3.9617	.5979	.9696	9.9866	50
20	.2476	9.3937	.2555	9.4074	3.9136	.5926	.9689	9.9863	40
30	.2504	9.3986	.2586	9.4127	3.8667	.5873	.9681	9.9859	30
40	.2532	9.4035	.2617	9.4178	3.8208	.5822	.9674	9.9856	20
50	.2560	9.4083	.2648	9.4230	3.7760	.5770	.9667	9.9853	10
15° 00′	.2588	9.4130	.2679	9.4281	3.7321	.5719	.9659	9.9849	**75° 00′**
10	.2616	9.4177	.2711	9.4331	3.6891	.5669	.9652	9.9846	50
20	.2644	9.4223	.2742	9.4381	3.6470	.5619	.9644	9.9843	40
30	.2672	9.4269	.2773	9.4430	3.6059	.5570	.9636	9.9839	30
40	.2700	9.4314	.2805	9.4479	3.5656	.5521	.9628	9.9836	20
50	.2728	9.4359	.2836	9.4527	3.5261	.5473	.9621	9.9832	10
16° 00′	.2756	9.4403	.2867	9.4575	3.4874	.5425	.9613	9.9828	**74° 00′**
10	.2784	9.4447	.2899	9.4622	3.4495	.5378	.9605	9.9825	50
20	.2812	9.4491	.2931	9.4669	3.4124	.5331	.9596	9.9821	40
30	.2840	9.4533	.2962	9.4716	3.3759	.5284	.9588	9.9817	30
40	.2868	9.4576	.2994	9.4762	3.3402	.5238	.9580	9.9814	20
50	.2896	9.4618	.3026	9.4808	3.3052	.5192	.9572	9.9810	10
17° 00′	.2924	9.4659	.3057	9.4853	3.2709	.5147	.9563	9.9806	**73° 00′**
10	.2952	9.4700	.3089	9.4898	3.2371	.5102	.9555	9.9802	50
20	.2979	9.4741	.3121	9.4943	3.2041	.5057	.9546	9.9798	40
30	.3007	9.4781	.3153	9.4987	3.1716	.5013	.9537	9.9794	30
40	.3035	9.4821	.3185	9.5031	3.1397	.4969	.9528	9.9790	20
50	.3062	9.4861	.3217	9.5075	3.1084	.4925	.9520	9.9786	10
18° 00′	.3090	9.4900	.3249	9.5118	3.0777	.4882	.9511	9.9782	**72° 00′**
	Value (Cosine)	Log (Cosine)	Value (Cotangent)	Log (Cotangent)	Value (Tangent)	Log (Tangent)	Value (Sine)	Log (Sine)	Degrees

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
18° 00′	.3090	9.4900	.3249	9.5118	3.0777	.4882	.9511	9.9782	**72° 00′**
10	.3118	9.4939	.3281	9.5161	3.0475	.4839	.9502	9.9778	50
20	.3145	9.4977	.3314	9.5203	3.0178	.4797	.9492	9.9774	40
30	.3173	9.5015	.3346	9.5245	2.9887	.4755	.9483	9.9770	30
40	.3201	9.5052	.3378	9.5287	2.9600	.4713	.9474	9.9765	20
50	.3228	9.5090	.3411	9.5329	2.9319	.4671	.9465	9.9761	10
19° 00′	.3256	9.5126	.3443	9.5370	2.9042	.4630	.9455	9.9757	**71° 00′**
10	.3283	9.5163	.3476	9.5411	2.8770	.4589	.9446	9.9752	50
20	.3311	9.5199	.3508	9.5451	2.8502	.4549	.9436	9.9748	40
30	.3338	9.5235	.3541	9.5491	2.8239	.4509	.9426	9.9743	30
40	.3365	9.5270	.3574	9.5531	2.7980	.4469	.9417	9.9739	20
50	.3393	9.5306	.3607	9.5571	2.7725	.4429	.9407	9.9734	10
20° 00′	.3420	9.5341	.3640	9.5611	2.7475	.4389	.9397	9.9730	**70° 00′**
10	.3448	9.5375	.3673	9.5650	2.7228	.4350	.9387	9.9725	50
20	.3475	9.5409	.3706	9.5689	2.6985	.4311	.9377	9.9721	40
30	.3502	9.5443	.3739	9.5727	2.6746	.4273	.9367	9.9716	30
40	.3529	9.5477	.3772	9.5766	2.6511	.4234	.9356	9.9711	20
50	.3557	9.5510	.3805	9.5804	2.6279	.4196	.9346	9.9706	10
21° 00′	.3584	9.5543	.3839	9.5842	2.6051	.4158	.9336	9.9702	**69° 00′**
10	.3611	9.5576	.3872	9.5879	2.5826	.4121	.9325	9.9697	50
20	.3638	9.5609	.3906	9.5917	2.5605	.4083	.9315	9.9692	40
30	.3665	9.5641	.3939	9.5954	2.5386	.4046	.9304	9.9687	30
40	.3692	9.5673	.3973	9.5991	2.5172	.4009	.9293	9.9682	20
50	.3719	9.5704	.4006	9.6028	2.4960	.3972	.9283	9.9677	10
22° 00′	.3746	9.5736	.4040	9.6064	2.4751	.3936	.9272	9.9672	**68° 00′**
10	.3773	9.5767	.4074	9.6100	2.4545	.3900	.9261	9.9667	50
20	.3800	9.5798	.4108	9.6136	2.4342	.3864	.9250	9.9661	40
30	.3827	9.5828	.4142	9.6172	2.4142	.3828	.9239	9.9656	30
40	.3854	9.5859	.4176	9.6208	2.3945	.3792	.9228	9.9651	20
50	.3881	9.5889	.4210	9.6243	2.3750	.3757	.9216	9.9646	10
23° 00′	.3907	9.5919	.4245	9.6279	2.3559	.3721	.9205	9.9640	**67° 00′**
10	.3934	9.5948	.4279	9.6314	2.3369	.3686	.9194	9.9635	50
20	.3961	9.5978	.4314	9.6348	2.3183	.3652	.9182	9.9629	40
30	.3987	9.6007	.4348	9.6383	2.2998	.3617	.9171	9.9624	30
40	.4014	9.6036	.4383	9.6417	2.2817	.3583	.9159	9.9618	20
50	.4041	9.6065	.4417	9.6452	2.2637	.3548	.9147	9.9613	10
24° 00′	.4067	9.6093	.4452	9.6486	2.2460	.3514	.9135	9.9607	**66° 00′**
10	.4094	9.6121	.4487	9.6520	2.2286	.3480	.9124	9.9602	50
20	.4120	9.6149	.4522	9.6553	2.2113	.3447	.9112	9.9596	40
30	.4147	9.6177	.4557	9.6587	2.1943	.3413	.9100	9.9590	30
40	.4173	9.6205	.4592	9.6620	2.1775	.3380	.9088	9.9584	20
50	.4200	9.6232	.4628	9.6654	2.1609	.3346	.9075	9.9579	10
25° 00′	.4226	9.6259	.4663	9.6687	2.1445	.3313	.9063	9.9573	**65° 00′**
10	.4253	9.6286	.4699	9.6720	2.1283	.3280	.9051	9.9567	50
20	.4279	9.6313	.4734	9.6752	2.1123	.3248	.9038	9.9561	40
30	.4305	9.6340	.4770	9.6785	2.0965	.3215	.9026	9.9555	30
40	.4331	9.6366	.4806	9.6817	2.0809	.3183	.9013	9.9549	20
50	.4358	9.6392	.4841	9.6850	2.0655	.3150	.9001	9.9543	10
26° 00′	.4384	9.6418	.4877	9.6882	2.0503	.3118	.8988	9.9537	**64° 00′**
10	.4410	9.6444	.4913	9.6914	2.0353	.3086	.8975	9.9530	50
20	.4436	9.6470	.4950	9.6946	2.0204	.3054	.8962	9.9524	40
30	.4462	9.6495	.4986	9.6977	2.0057	.3023	.8949	9.9518	30
40	.4488	9.6521	.5022	9.7009	1.9912	.2991	.8936	9.9512	20
50	.4514	9.6546	.5059	9.7040	1.9768	.2960	.8923	9.9505	10
27° 00′	.4540	9.6570	.5095	9.7072	1.9626	.2928	.8910	9.9499	**63° 00′**
	Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees

Degrees	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
27° 00′	.4540	9.6570	.5095	9.7072	1.9626	.2928	.8910	9.9499	**63° 00′**
10	.4566	9.6595	.5132	9.7103	1.9486	.2897	.8897	9.9492	50
20	.4592	9.6620	.5169	9.7134	1.9347	.2866	.8884	9.9486	40
30	.4617	9.6644	.5206	9.7165	1.9210	.2835	.8870	9.9479	30
40	.4643	9.6668	.5243	9.7196	1.9074	.2804	.8857	9.9473	20
50	.4669	9.6692	.5280	9.7226	1.8940	.2774	.8843	9.9466	10
28° 00′	.4695	9.6716	.5317	9.7257	1.8807	.2743	.8829	9.9459	**62° 00′**
10	.4720	9.6740	.5354	9.7287	1.8676	.2713	.8816	9.9453	50
20	.4746	9.6763	.5392	9.7317	1.8546	.2683	.8802	9.9446	40
30	.4772	9.6787	.5430	9.7348	1.8418	.2652	.8788	9.9439	30
40	.4797	9.6810	.5467	9.7378	1.8291	.2622	.8774	9.9432	20
50	.4823	9.6833	.5505	9.7408	1.8165	.2592	.8760	9.9425	10
29° 00′	.4848	9.6856	.5543	9.7438	1.8040	.2562	.8746	9.9418	**61° 00′**
10	.4874	9.6878	.5581	9.7467	1.7917	.2533	.8732	9.9411	50
20	.4899	9.6901	.5619	9.7497	1.7796	.2503	.8718	9.9404	40
30	.4924	9.6923	.5658	9.7526	1.7675	.2474	.8704	9.9397	30
40	.4950	9.6946	.5696	9.7556	1.7556	.2444	.8689	9.9390	20
50	.4975	9.6968	.5735	9.7585	1.7437	.2415	.8675	9.9383	10
30° 00′	.5000	9.6990	.5774	9.7614	1.7321	.2386	.8660	9.9375	**60° 00′**
10	.5025	9.7012	.5812	9.7644	1.7205	.2356	.8646	9.9368	50
20	.5050	9.7033	.5851	9.7673	1.7090	.2327	.8631	9.9361	40
30	.5075	9.7055	.5890	9.7701	1.6977	.2299	.8616	9.9353	30
40	.5100	9.7076	.5930	9.7730	1.6864	.2270	.8601	9.9346	20
50	.5125	9.7097	.5969	9.7759	1.6753	.2241	.8587	9.9338	10
31° 00′	.5150	9.7118	.6009	9.7788	1.6643	.2212	.8572	9.9331	**59° 00′**
10	.5175	9.7139	.6048	9.7816	1.6534	.2184	.8557	9.9323	50
20	.5200	9.7160	.6088	9.7845	1.6426	.2155	.8542	9.9315	40
30	.5225	9.7181	.6128	9.7873	1.6319	.2127	.8526	9.9308	30
40	.5250	9.7201	.6168	9.7902	1.6212	.2098	.8511	9.9300	20
50	.5275	9.7222	.6208	9.7930	1.6107	.2070	.8496	9.9292	10
32° 00′	.5299	9.7242	.6249	9.7958	1.6003	.2042	.8480	9.9284	**58° 00′**
10	.5324	9.7262	.6289	9.7986	1.5900	.2014	.8465	9.9276	50
20	.5348	9.7282	.6330	9.8014	1.5798	.1986	.8450	9.9268	40
30	.5373	9.7302	.6371	9.8042	1.5697	.1958	.8434	9.9260	30
40	.5398	9.7322	.6412	9.8070	1.5597	.1930	.8418	9.9252	20
50	.5422	9.7342	.6453	9.8097	1.5497	.1903	.8403	9.9244	10
33° 00′	.5446	9.7361	.6494	9.8125	1.5399	.1875	.8387	9.9236	**57° 00′**
10	.5471	9.7380	.6536	9.8153	1.5301	.1847	.8371	9.9228	50
20	.5495	9.7400	.6577	9.8180	1.5204	.1820	.8355	9.9219	40
30	.5519	9.7419	.6619	9.8208	1.5108	.1792	.8339	9.9211	30
40	.5544	9.7438	.6661	9.8235	1.5013	.1765	.8323	9.9203	20
50	.5568	9.7457	.6703	9.8263	1.4919	.1737	.8307	9.9194	10
34° 00′	.5592	9.7476	.6745	9.8290	1.4826	.1710	.8290	9.9186	**56° 00′**
10	.5616	9.7494	.6787	9.8317	1.4733	.1683	.8274	9.9177	50
20	.5640	9.7513	.6830	9.8344	1.4641	.1656	.8258	9.9169	40
30	.5664	9.7531	.6873	9.8371	1.4550	.1629	.8241	9.9160	30
40	.5688	9.7550	.6916	9.8398	1.4460	.1602	.8225	9.9151	20
50	.5712	9.7568	.6959	9.8425	1.4370	.1575	.8208	9.9142	10
35° 00′	.5736	9.7586	.7002	9.8452	1.4281	.1548	.8192	9.9134	**55° 00′**
10	.5760	9.7604	.7046	9.8479	1.4193	.1521	.8175	9.9125	50
20	.5783	9.7622	.7089	9.8506	1.4106	.1494	.8158	9.9116	40
30	.5807	9.7640	.7133	9.8533	1.4019	.1467	.8141	9.9107	30
40	.5831	9.7657	.7177	9.8559	1.3934	.1441	.8124	9.9098	20
50	.5854	9.7675	.7221	9.8586	1.3848	.1414	.8107	9.9089	10
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	.1387	.8090	9.9080	**54° 00′**
	Value	Log	Value	Log	Value	Log	Value	Log	Degrees
	Cosine		Cotangent		Tangent		Sine		

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	.1387	.8090	9.9080	**54° 00′**
10	.5901	9.7710	.7310	9.8639	1.3680	.1361	.8073	9.9070	50
20	.5925	9.7727	.7355	9.8666	1.3597	.1334	.8056	9.9061	40
30	.5948	9.7744	.7400	9.8692	1.3514	.1308	.8039	9.9052	30
40	.5972	9.7761	.7445	9.8718	1.3432	.1282	.8021	9.9042	20
50	.5995	9.7778	.7490	9.8745	1.3351	.1255	.8004	9.9033	10
37° 00′	.6018	9.7795	.7536	9.8771	1.3270	.1229	.7986	9.9023	**53° 00′**
10	.6041	9.7811	.7581	9.8797	1.3190	.1203	.7969	9.9014	50
20	.6065	9.7828	.7627	9.8824	1.3111	.1176	.7951	9.9004	40
30	.6088	9.7844	.7673	9.8850	1.3032	.1150	.7934	9.8995	30
40	.6111	9.7861	.7720	9.8876	1.2954	.1124	.7916	9.8985	20
50	.6134	9.7877	.7766	9.8902	1.2876	.1098	.7898	9.8975	10
38° 00′	.6157	9.7893	.7813	9.8928	1.2799	.1072	.7880	9.8965	**52° 00′**
10	.6180	9.7910	.7860	9.8954	1.2723	.1046	.7862	9.8955	50
20	.6202	9.7926	.7907	9.8980	1.2647	.1020	.7844	9.8945	40
30	.6225	9.7941	.7954	9.9006	1.2572	.0994	.7826	9.8935	30
40	.6248	9.7957	.8002	9.9032	1.2497	.0968	.7808	9.8925	20
50	.6271	9.7973	.8050	9.9058	1.2423	.0942	.7790	9.8915	10
39° 00′	.6293	9.7989	.8098	9.9084	1.2349	.0916	.7771	9.8905	**51° 00′**
10	.6316	9.8004	.8146	9.9110	1.2276	.0890	.7753	9.8895	50
20	.6338	9.8020	.8195	9.9135	1.2203	.0865	.7735	9.8884	40
30	.6361	9.8035	.8243	9.9161	1.2131	.0839	.7716	9.8874	30
40	.6383	9.8050	.8292	9.9187	1.2059	.0813	.7698	9.8864	20
50	.6406	9.8066	.8342	9.9212	1.1988	.0788	.7679	9.8853	10
40° 00′	.6428	9.8081	.8391	9.9238	1.1918	.0762	.7660	9.8843	**50° 00′**
10	.6450	9.8096	.8441	9.9264	1.1847	.0736	.7642	9.8832	50
20	.6472	9.8111	.8491	9.9289	1.1778	.0711	.7623	9.8821	40
30	.6494	9.8125	.8541	9.9315	1.1708	.0685	.7604	9.8810	30
40	.6517	9.8140	.8591	9.9341	1.1640	.0659	.7585	9.8800	20
50	.6539	9.8155	.8642	9.9366	1.1571	.0634	.7566	9.8789	10
41° 00′	.6561	9.8169	.8693	9.9392	1.1504	.0608	.7547	9.8778	**49° 00′**
10	.6583	9.8184	.8744	9.9417	1.1436	.0583	.7528	9.8767	50
20	.6604	9.8198	.8796	9.9443	1.1369	.0557	.7509	9.8756	40
30	.6626	9.8213	.8847	9.9468	1.1303	.0532	.7490	9.8745	30
40	.6648	9.8227	.8899	9.9494	1.1237	.0506	.7470	9.8733	20
50	.6670	9.8241	.8952	9.9519	1.1171	.0481	.7451	9.8722	10
42° 00′	.6691	9.8255	.9004	9.9544	1.1106	.0456	.7431	9.8711	**48° 00′**
10	.6713	9.8269	.9057	9.9570	1.1041	.0430	.7412	9.8699	50
20	.6734	9.8283	.9110	9.9595	1.0977	.0405	.7392	9.8688	40
30	.6756	9.8297	.9163	9.9621	1.0913	.0379	.7373	9.8676	30
40	.6777	9.8311	.9217	9.9646	1.0850	.0354	.7353	9.8665	20
50	.6799	9.8324	.9271	9.9671	1.0786	.0329	.7333	9.8653	10
43° 00′	.6820	9.8338	.9325	9.9697	1.0724	.0303	.7314	9.8641	**47° 00′**
10	.6841	9.8351	.9380	9.9722	1.0661	.0278	.7294	9.8629	50
20	.6862	9.8365	.9435	9.9747	1.0599	.0253	.7274	9.8618	40
30	.6884	9.8378	.9490	9.9772	1.0538	.0228	.7254	9.8606	30
40	.6905	9.8391	.9545	9.9798	1.0477	.0202	.7234	9.8594	20
50	.6926	9.8405	.9601	9.9823	1.0416	.0177	.7214	9.8582	10
44° 00′	.6947	9.8418	.9657	9.9848	1.0355	.0152	.7193	9.8569	**46° 00′**
10	.6967	9.8431	.9713	9.9874	1.0295	.0126	.7173	9.8557	50
20	.6988	9.8444	.9770	9.9899	1.0235	.0101	.7153	9.8545	40
30	.7009	9.8457	.9827	9.9924	1.0176	.0076	.7133	9.8532	30
40	.7030	9.8469	.9884	9.9949	1.0117	.0051	.7112	9.8520	20
50	.7050	9.8482	.9942	9.9975	1.0058	.0025	.7092	9.8507	10
45° 00′	.7071	9.8495	1.0000	.0000	1.0000	.0000	.7071	9.8495	**45° 00′**
	Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees

TABLE III

SQUARES AND SQUARE ROOTS

SQUARES AND SQUARE ROOTS

(Moving the decimal point *one* place in N requires a corresponding move of *two* places in N^2.)

N	N^2 0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0001	.0004	.0009	.0016	.0025	.0036	.0049	.0064	.0081
0.1	.0100	.0121	.0144	.0169	.0196	.0225	.0256	.0289	.0324	.0361
0.2	.0400	.0441	.0484	.0529	.0576	.0625	.0676	.0729	.0784	.0841
0.3	.0900	.0961	.1024	.1089	.1156	.1225	.1296	.1369	.1444	.1521
0.4	.1600	.1681	.1764	.1849	.1936	.2025	.2116	.2209	.2304	.2401
0.5	.2500	.2601	.2704	.2809	.2916	.3025	.3136	.3249	.3364	.3481
0.6	.3600	.3721	.3844	.3969	.4096	.4225	.4356	.4489	.4624	.4761
0.7	.4900	.5041	.5184	.5329	.5476	.5625	.5776	.5929	.6084	.6241
0.8	.6400	.6561	.6724	.6889	.7056	.7225	.7396	.7569	.7744	.7921
0.9	.8100	.8281	.8464	.8649	.8836	.9025	.9216	.9409	.9604	.9801
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.752	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.673	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91

SQUARES AND SQUARE ROOTS

(Moving the decimal point *one* place in N requires a corresponding move of *two* places in N^2.)

			2							
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	.91
	6.0	26.1	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
	27.04	27.1	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
	28.0	28.	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
	29.1	29.2	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14
	30.2	30.3	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
	31.3	31.4	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
	32.4	32.6	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
	33.64	33.7	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
	34.8	34.9	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
	36.0	36.1	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
	37.2	37.3	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
	38.44	38.5	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
	39.6	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
	40.96	41.0	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
	46.24	46.3	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
	7.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
	9.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
	2.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
	4.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
	5.61	65.77	5.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
	7.24	7.40	7.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
	8.89	9.06	9.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
	0.56	0.73	0.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
	2.25	2.42	2.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
	3.96	4.13	4.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
	5.69	5.86	6.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
	7.44	77.62	7.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
	9.21	9.39	9.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
	1.00	1.18	1.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
	2.81	2.99	3.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
	4.64	4.82	5.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
	6.49	6.68	6.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
	8.36	8.55	8.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
	0.25	0.44	0.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
	2.16	2.35	2.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
	4.09	4.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
	6.04	6.24	6.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
	8.01	8.21	8.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

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